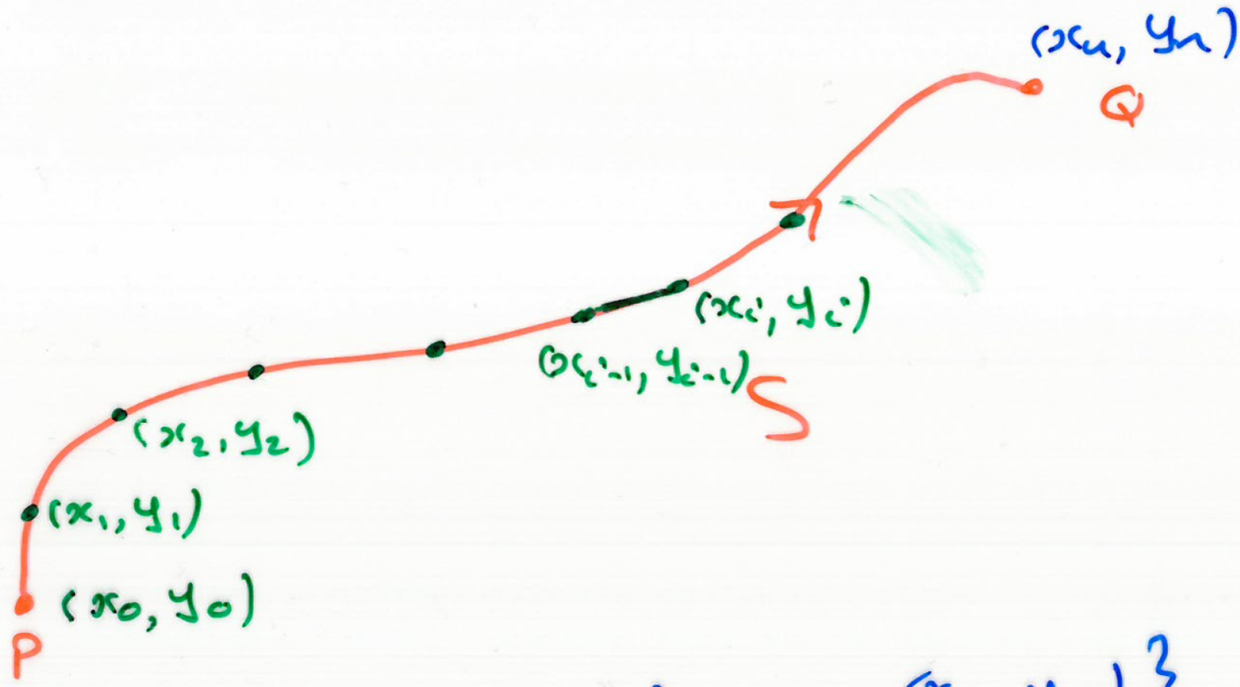
More formally

$$\int_S A(x,y)\,dx + B(x,y)\,dy =$$

$$\lim_{\|P\|\to 0} \sum A(x_i, y_i)(x_i - x_{i-1}) + B(x_i, y_i)(y_i - y_{i-1})$$



where $P = \{(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)\}$ is a sequence of points on S with (x_0, y_0) the initial point and (x_n, y_n) the final point of S.

and

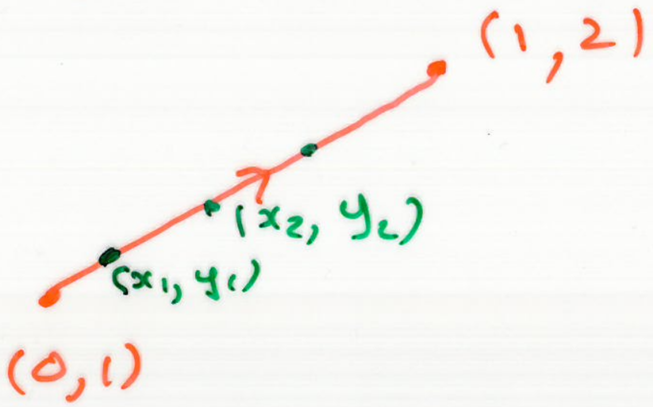
$$\|P\| = \max_{1 \le i \le n} \|(x_i, y_i) - (x_{i-1}, y_{i-1})\|$$

with $\|(x,y)\| = \sqrt{x^2+y^2}$

<u>Example</u> Let S be the line segment from $(0,1)$ to $(1,2)$. Evaluate

$$L = \int_S (x^2-y)\,dx + (y^2+x)\,dy .$$

<u>Sol^n</u> The line $y = x+1$



passes through $(0,1)$ and $(1,2)$.

$$P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)\}$$

$$= \{(x_0, x_0+1), (x_1, x_1+1), \cdots, (x_n, x_n+1)\}$$

$$L = \lim_{\|P\| \to 0} \sum_{i=1}^{n} A(x_i, y_i)(x_i - x_{i-1}) + B(x_i, y_i)(y_i - y_{i-1})$$

$$= \lim_{\|P\| \to 0} \sum_{i=1}^{n} (x_i^2 - x_i - 1)(x_i - x_{i-1}) + ((x_i+1)^2 + x_i)(x_i - x_{i-1})$$

$$= \int_0^1 (x^2 - x - 1 + (x+1)^2 + x) \, dx$$

$$= \int_0^1 (2x^2 + 2x) \, dx$$

$$= \cdots$$

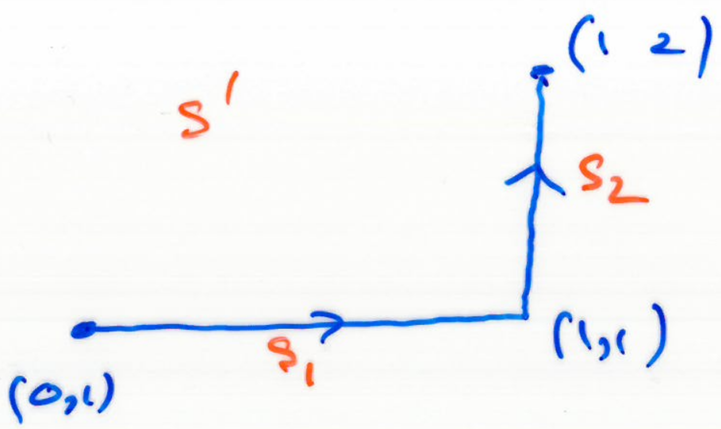
$$= \frac{5}{3}$$

Evaluate

$$L' = \int_{S'} (x^2 - y)\,dx + (y^2 + x)\,dy$$

where S' is the line from $(0,1)$ to $(1,1)$ followed by the line from $(1,1)$ to $(1,2)$.

Sol^n



$$L' = \int_{S_1} (x^2 - y)\,dx + (y^2 + x)\,dy$$

$$+ \int_{S_2} (x^2 - y)\,dx + (y^2 + x)\,dy$$

The line $y = 1$ contains S_1

" " $x = 1$ " S_2

$L' =$

$$\lim_{\|P_1\|} \sum_{i=1}^{n} (x_i^2 - 1)(x_i - x_{i-1}) + (1^2 + x_i).0$$

$+$

$$\lim_{\|P_2\|} \sum_{i=1}^{n} (1^2 - y_i).0 + (y_i^2 + 1)(y_i - y_{i-1})$$

$$= \int_0^1 x^2 - 1 \quad dx + \int_1^2 y^2 + 1 \quad dy$$

$= \cdots$

$= -2.$

Example work is represented by the 1-form

$$\omega = (3x - 4y + 2z)\,dx + (4x + 2y - 3z^2)\,dy + (2xz - 4y^2 + z^3)\,dz.$$

Find the work done in moving a particle once around the following ellipse in the xy-plane, in the anti-clockwise direction.