Third Test: this Wednesday

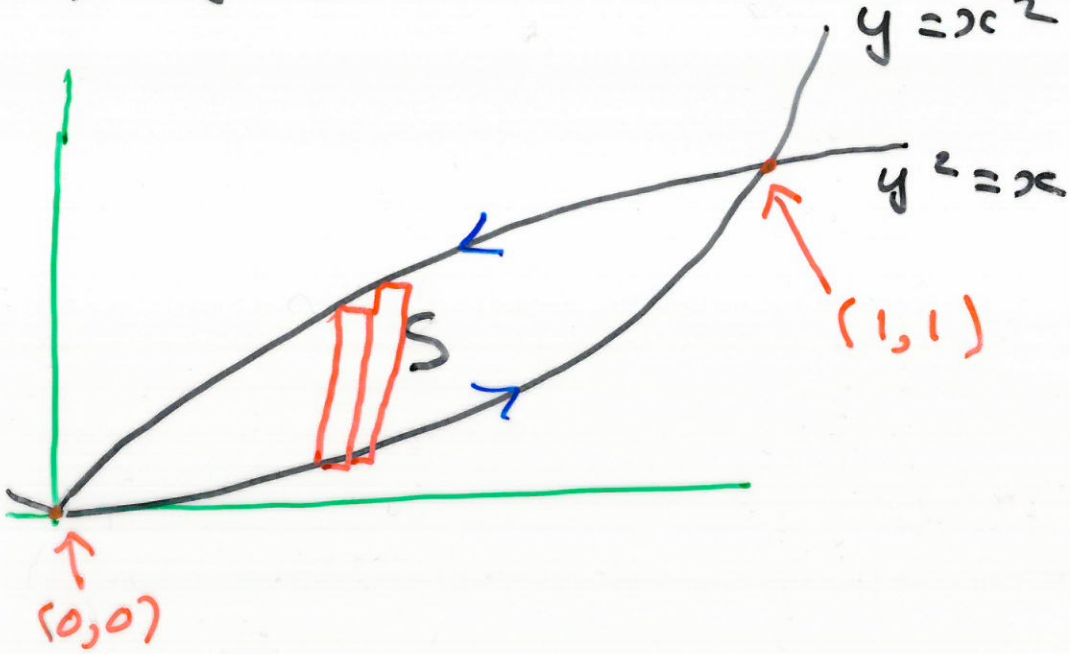
Problem 10.6

Verify Green's Theorem in the plane for

$$\omega = (2xy - x^2)\,dx + (x + y^2)\,dy$$

and S the region bounded by $y = x^2$, $y^2 = x$.

Sol^n



Need to verify

$$\int_{\partial S} \omega = \int_S d\omega$$

$$LHS = \int_{\partial S} \omega$$

$$= \int_{\partial S} (2xy - x^2)\,dx + (x + y^2)\,dy$$

$y = x^2$, $x = t$, $y = t^2$, $dx = dt$, $dy = 2t\,dt$

$$= \int_0^1 \left(2t^3 - t^2 + 2(t + t^4)\,t\right) dt$$

$y^2 = x$, $y = t$, $x = t^2$, $dy = dt$, $dx = 2t\,dt$

$$+ \int_1^0 2(2t^3 - t^4)\,t\,dt + (t^2 + t^2)\,dt$$

$$= \frac{7}{6} - \frac{17}{15} = \frac{1}{30}.$$

$$RHS = \int_S d\omega$$

$$= \int_S d\Big((2xy - x^2)\, dx + (x+y^2)\, dy \Big)$$

$$= \int_S 2x\, dy \wedge dx + dx \wedge dy$$

$$= \int_S (1-2x)\, dx \wedge dy$$

$$= \int_{x=0}^{1} \Big(\int_{y=x^2}^{y=\sqrt{x}} 1 - 2x \, dy \Big) dx$$

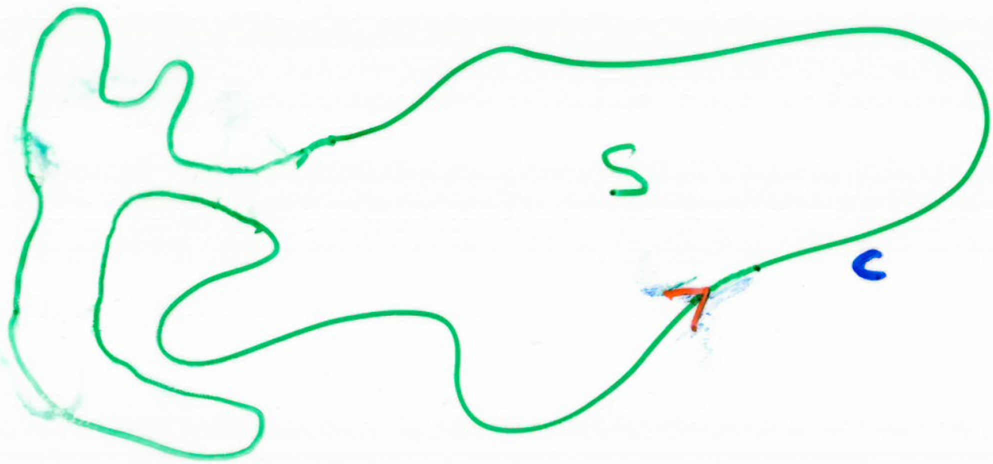
$$= \int_{x=0}^{1} y - 2xy \Big|_{y=x^2}^{y=\sqrt{x}} dx$$

$$= \int_0^1 x^{\frac{1}{2}} - 2x^{\frac{3}{2}} - x^2 + 2x^3 \, dx$$

$= \ldots$

$= \frac{1}{30}$.

Problem Show that the area of the region S bounded by a simple closed curve $C = \partial S$ in the xy-plane



is $\frac{1}{2}\int_C x\,dy - y\,dx$.

Sol^n

$$\frac{1}{2}\int_C x\,dy - y\,dx$$

$$= \frac{1}{2}\int_S d(x\,dy - y\,dx)$$

Stokes' formula

$$= \frac{1}{2}\int_S dx \wedge dy - dy \wedge dx$$

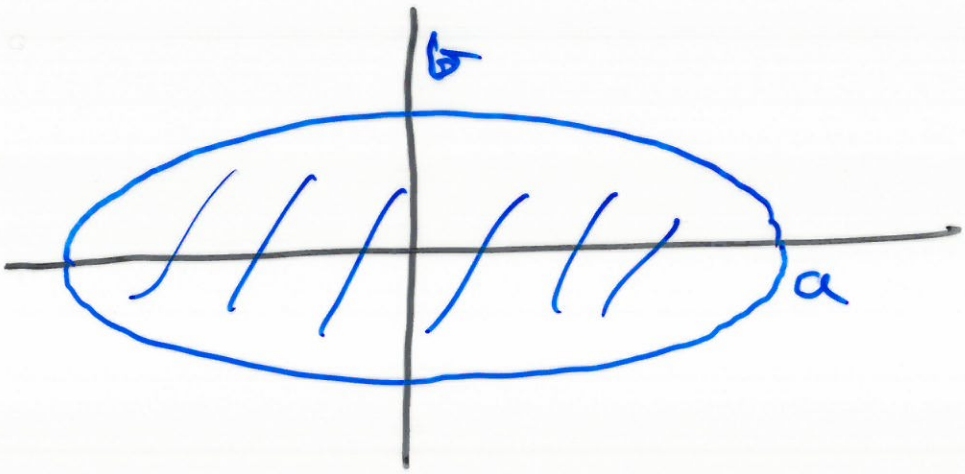
$$= \int_S dx \wedge dy$$

$= \text{area of } S$ (by definition).

Example Find the area of the region S bounded by the ellipse

$$x = a\cos t, \quad y = b\sin t$$

Solⁿ



From the preceding example

required area =

$$\frac{1}{2}\int_C x\,dy - y\,dx$$

$$= \frac{1}{2} \int_0^{2\pi} a\cos(t)\, b\cos(t)\, dt + ab\sin(t)\sin(t)\, dt$$

$$= \frac{ab}{2} \int_0^{2\pi} \cos^2(t) + \sin^2(t)\, dt$$

$$= \frac{ab}{2} \int_0^{2\pi} dt$$

$$= \frac{ab}{2} \cdot 2\pi$$

$$= \underline{\underline{ab\pi}}$$