Example Integrate the 0-form $\omega = 3x^2+4$ on ∂S where $S = [2,1] \cup [3,4]$.

Soln

$$\int_{\partial S} \omega \overset{\uparrow \text{definition}}{=} \int_{\partial [2,1]} \omega + \int_{\partial [3,4]} \omega$$

$$= \omega(1) - \omega(2) + \omega(4) - \omega(3)$$

$$= 7 - 16 + 52 - 31$$

$$= 12.$$

Background reading:

"Advanced calculus: a differential forms approach"

by Harold M. Edwards.

Also: Spivak's book on manifolds.

Stokes' formula

$$\int_{\partial S} \omega = \int_S d\omega$$

$n = 1$ variables

$p = 0$, ω is a 0-form

Differential 1-forms in 1 variable

A differential 1-form is a function of the form

$$\omega = f(x)\, h$$

which inputs two numbers $x, h \in \mathbb{R}$, and returns the number $f(x)\, h$, where $f(x)$ is some function.

Example Evaluate the 1-form

$$\omega = (x^2+6)\,h$$

at $x=2$, $h=0.5$.

Soln

$$(2^2+6)\,0.5 = 5$$

Notation: We usually denote the 1-form

$$\omega = f(x)\,h$$

by

$$\omega = f(x)\,dx$$

Example Evaluate the 1-form

$$\omega = \sin(x)\,dx$$

at $x = \frac{\pi}{2}$, $dx = 0.25$.

Soln

$$\sin\left(\frac{\pi}{2}\right) \times 0.25 = 0.25$$

Defn Given a 1-form

$$\omega = f(x)\,dx$$

and an oriented interval $S = [a, b]$ we define the integral as

$$\int_S \omega = \int_a^b f(x)\,dx$$

explained in 1st year

informally: $\int_a^b f(x)\,dx$ is the area between the curve $y = f(x)$ and the x-axis from a to b, where if $b > a$ areas above the x-axis are regarded as positive, and areas below

the x-axis are regarded as negative.

Problem A fundraising project has daily expenditure of $10 000. The rate of contributions at time t is modelled by

$$C(t) = -100t^2 + 20\,000.$$

What net proceeds can be expected.

Soln project runs until

$$C(t) \leq 10\,000.$$

$$-100t^2 + 20000 = 10\,000$$

$$100t^2 = 10\,000$$

$$t = 10.$$

The project will run from $t = 0$ to $t = 10$.

Contributions are modelled by the 1-form

$$u = (-100t^2 + 20\,000)\,dt$$

Expenditure is modelled by the 1-form

$$v = -10\,000\,dt$$

The net rate of income is modelled by the 1-form

$$w = u + v = (-100t^2 + 10\,000)\,dt$$

The project can be expected to make

$$\int_S w$$

where $S = [0, 10]$.

$$\int_S \omega \overset{\uparrow \text{def}}{=} \int_0^{10} -100t^2 + 10\,000\,dt$$

$$= \left. \frac{-100t^3}{3} + 10\,000t \right|_0^{10}$$

$$= \$\ 66\,666.67$$