Divergence

Given a vector field

$F = F_1 \underline{\underline{i}} + F_2 \underline{\underline{j}} + F_3 \underline{\underline{k}}$

on $\mathbb{R}^3$ we define the associated flux 2-form

$$\omega = F_3\, dx \wedge dy + F_1\, dy \wedge dz + F_2\, dz \wedge dx$$

The exterior derivative of ω is a 3-form

$$d\omega = A\, dx \wedge dy \wedge dz$$

$$\boxed{\begin{array}{l}\textbf{Definition} \text{ we define the } \textbf{divergence} \text{ of } F \text{ to be the function} \\ \qquad \mathrm{div}(F) = A\end{array}}$$

<u>Example</u> Consider

$$F = xz\, \underline{\underline{i}} - y^2\, \underline{\underline{j}} + 2x^2 y\, \underline{\underline{k}}\,.$$

Let's find div(F).

$$\omega = 2x^2y\, dx \wedge dy + xz\, dy \wedge dz - y^2\, dz \wedge dx$$

$$d\omega = z\, dx \wedge dy \wedge dz - 2y\, dy \wedge dz \wedge dx$$

$$= z\, dx \wedge dy \wedge dz - 2y\, dx \wedge dy \wedge dz$$

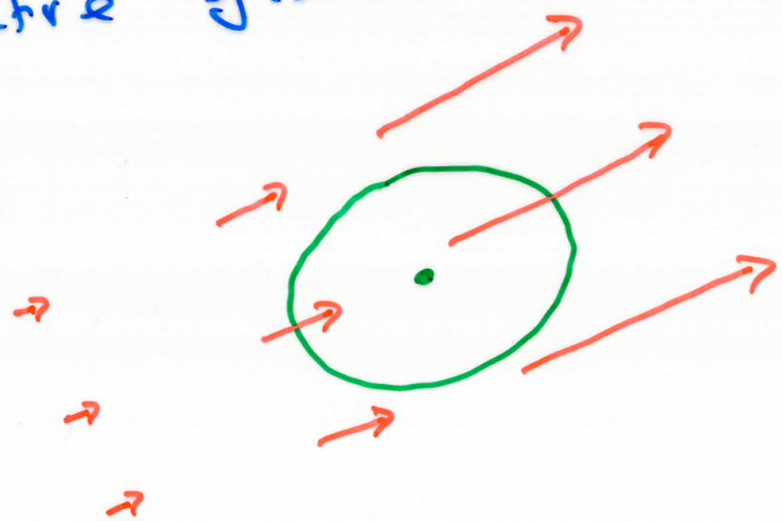
$$= (z - 2y)\, dx \wedge dy \wedge dz$$

$$\boxed{\operatorname{div}(F) = z - 2y}$$

Interpretation

Let F represent the flow of a fluid in $\mathbb{R}^3$.

Place a small ball with centre fixed at (x, y, z).

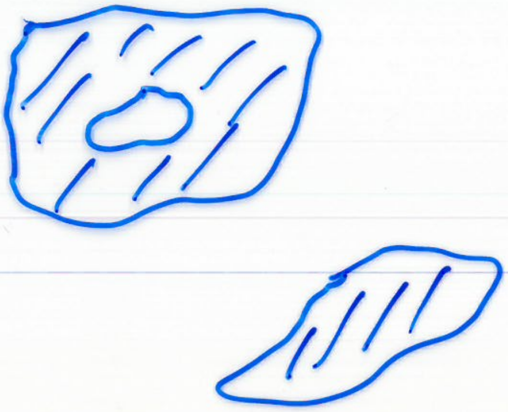


Fluid flows into the ball and out of the ball.

The difference is measured by the number

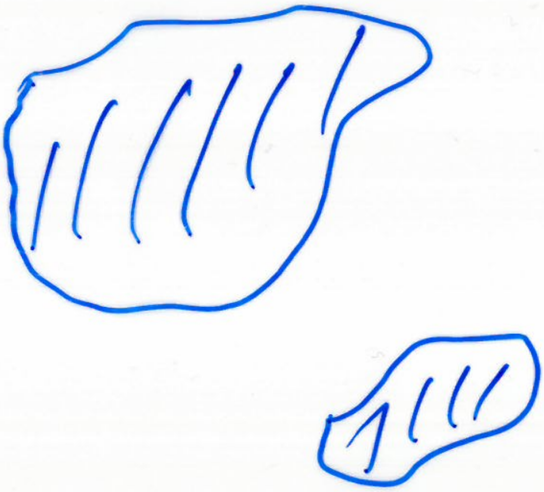
$$\operatorname{div}(F)(x, y, z).$$

Regions in plane



not connected

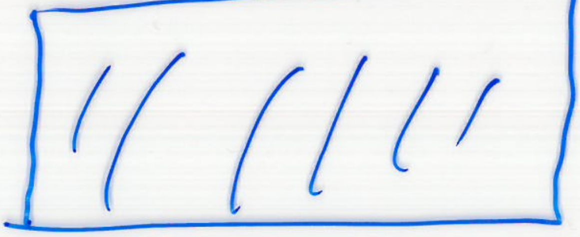
not simply connected



not connected

connected

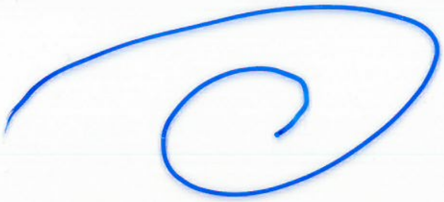
not simply connected



connected

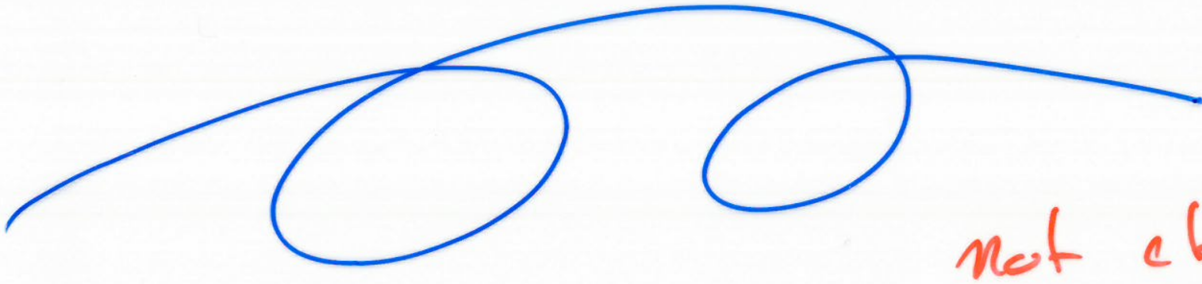
simply connected.

Curves



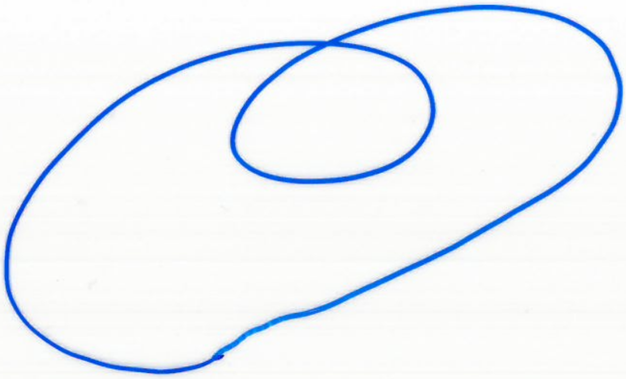
not closed

simple curve



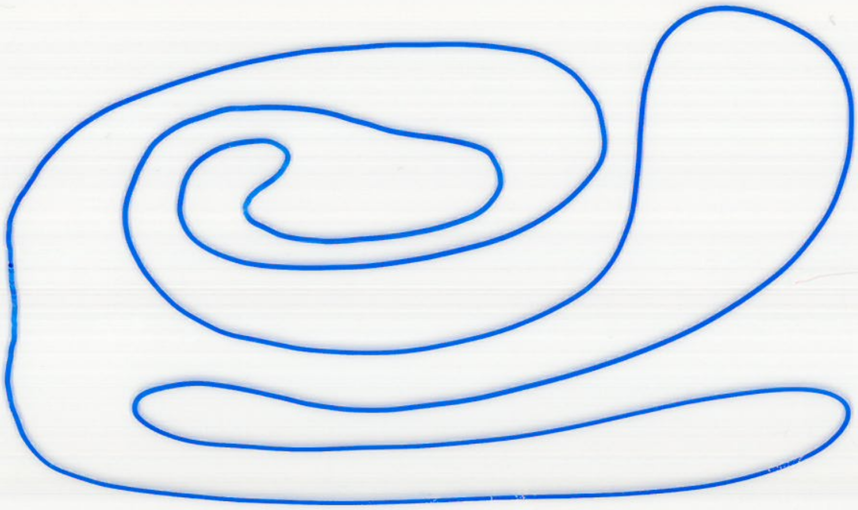
not closed

not a simple curve



closed

not a simple curve



closed

simple curve

Green's Theorem in the plane

Let $P = P(x,y)$, $Q = Q(x,y)$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$ be single valued and continuous in a simple connected region S bounded by a simple closed curve C. Then

$$(*) \quad \oint_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

where C is travelled in the positive direction. [Page 246].

i.e. for $\omega = P\,dx + Q\,dy$

$$\int_{\partial S} \omega = \int_S d\omega$$

On page 251 you'll find Stokes' theorem, which is a generalisation of Green's Theorem to the case where S is a simply connected region on a surface.

Equation (*) becomes a bit more involved. Alternatively (*) can be written

$$\int_{\partial S} \omega = \int_S d\omega .$$

Divergence Theorem

Let F be a vector field that is continuously differentiable on a closed space region V bounded by a smooth surface S. Then

$$\iiint_V \nabla \cdot F \, dv = \iint_S F \cdot \underline{n} \, dS$$

where $\underline{n}$ is an outward pointing normal to S.

i.e. for an 3-form $\omega = F \cdot \underline{n}$

$$\int_V d\omega = \int_{\partial V} \omega \, .$$