Third Test : Wed 28 November

A function $\phi(x, y, z)$ is harmonic if it is continuous, and if its average over any ball in its domain of definition

$$D = \{ \underline{x} \in \mathbb{R}^3 : \| \underline{x} - \underline{c} \| \leq r \}$$

is equal to its value at the centre of the ball :

$$\phi(\underline{c}) = \frac{3}{4\pi r^3} \int_D \phi(x, y, z) \, dx \wedge dy \wedge dz$$

<u>Theorem</u> ϕ is harmonic if and only if

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

Electrostatics

concerns a charge density $\rho = \rho(x,y,z)$, and a potential $\phi(x,y,z)$.

It is postulated that

$$\varepsilon\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}\right)+\rho=0 \qquad (**)$$

i.e. the potential is harmonic at places where there is no charge.

Equation $(**)$ is called Poisson's Equation.

ε is called the dielectric constant.

We can consider the 1-form

$$E = -d\phi = E_1 dx + E_2 dy + E_3 dz$$

which is called the electric force field. It describes the work needed to displace a charge.

The 2-form

$$D = -\varepsilon\left(\frac{\partial\phi}{\partial x} dy \wedge dz + \frac{\partial\phi}{\partial y} dz \wedge dx + \frac{\partial\phi}{\partial z} dx \wedge dy\right)$$

is called the electric displacement.

Poisson's Equation (**) can be expressed as

$$\boxed{\rho\, dx \wedge dy \wedge dz = dD}$$

Electrodynamics

A moving particle is acted on by forces other than those of electrostatics — namely magnetic force.

Faraday postulated (in our language!) that magnetic forces are described by a 2-form

$$B = B_1\, dx \wedge dy + B_2\, dy \wedge dz + B_3\, dz \wedge dx$$

and that the relationship to electric forces is governed by

$$d\,(E\wedge dt + B) = 0 .$$

Here

$$E\wedge dt = E_1\, dx\wedge dt + E_2\, dy\wedge dt + E_3\, dz\wedge dt$$

The 2-form $E\wedge dt + B$ is called the <u>electromagnetic field</u>. The charge and its motion are described by a 3-form

$$\begin{aligned} J = \rho\, dx\wedge dy\wedge dz &- j_1\, dy\wedge dz\wedge dt \\ &- j_2\, dz\wedge dx\wedge dt \\ &- j_3\, dx\wedge dy\wedge dt \end{aligned}$$

called the <u>moving</u> <u>charge</u> (or <u>current</u>)

Maxwell's Equations

Electromagnetism is a mathematical theory based on the follows definitions:

1) Electromagnetic field $= E \wedge dt + B$

(2-form)

2) Moving charge $= J$

(3-form)

3) $D = -(\varepsilon E_1 \, dy \wedge dz + \varepsilon E_2 \, dz \wedge dx + \varepsilon E_3 \, dx \wedge dy)$

(2-form)

4) $H = \frac{1}{\mu} B_1 \, dx + \frac{1}{\mu} B_2 \, dy + \frac{1}{\mu} B_3 \, dz$

(1-form)

μ = magnetic permeability

The theory is described by the following equations:

- $d(E \wedge dt + B) = 0$ Faraday's Law

- $dJ = 0$ Gauss's Law

- $d(D - H \wedge dt) = J$ Ampère / Maxwell's Law