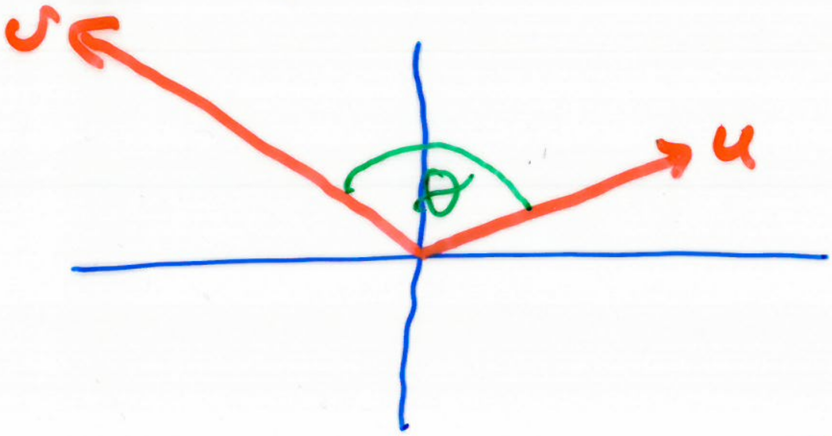
Dot products of vectors

Given two vectors

$$u = (u_1, u_2) \in \mathbb{R}^2$$
$$v = (v_1, v_2) \in \mathbb{R}^2$$

in the plane.



we define their <u>dot product</u> to be the number

$$u.v = u_1 v_1 + u_2 v_2 .$$

<u>Example</u> if $u = (2,3)$, $v = (4,5)$

then $u.v = 2.4 + 3.5 = 23$

We define the <u>length</u> of u to be

$$|u| = \sqrt{u_1^2 + u_2^2} = \sqrt{u.u}$$

It's easy to prove

<u>Theorem</u> $u.v = |u|.|v|.\cos(\theta)$

In particular, u and v are perpendicular to each other if and only if $u.v = 0$.

For two vectors

$$u = (u_1, u_2, u_3) \in \mathbb{R}^3$$

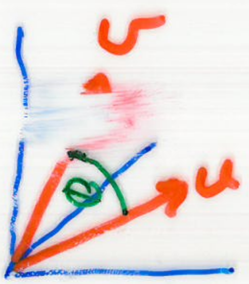
$$v = (v_1, v_2, v_3) \in \mathbb{R}^3$$

we define their <u>dot product</u>

$$u.v = u_1v_1 + u_2v_2 + u_3v_3$$

Also we define

$$|u| = \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{u.u}$$



<u>Theorem</u>

$u.v = |u|.|v|.\cos\theta$

so u, v are at right angles iff $u.v = 0$.

Div, Grad, Curl and all that

Gradient

Let $\phi(x,y,z)$ be a real valued differentiable function

$$\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}.$$

The gradient of ϕ is defined as

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\underline{\underline{i}} + \frac{\partial\phi}{\partial y}\underline{\underline{j}} + \frac{\partial\phi}{\partial z}\underline{\underline{k}}$$

where

$\underline{\underline{i}} = (1,0,0)$

$\underline{\underline{j}} = (0,1,0)$

$\underline{\underline{k}} = (0,0,1)$.

Example $\quad \phi(x,y,z) = x^2+y^2+z^2$

$$\text{grad}\,\phi = \nabla\phi = 2x\,\underline{\underline{i}} + 2y\,\underline{\underline{j}} + 2z\,\underline{\underline{k}}$$

$$= (2x, 2y, 2z).$$

We can think of the gradient grad ϕ as the derivative of a 0-form

$$\nabla \phi \simeq d\phi$$

<u>interpretation of the gradient (or of the total derivative) of a 0-form in 3-variables)</u>

Consider a surface S defined by an equation

$$\phi(x, y, z) = k \quad (k \text{ constant})$$

<u>Example</u> Let $\phi(x, y, z) = x^2 + y^2 + z^2$

Let $k = 9$

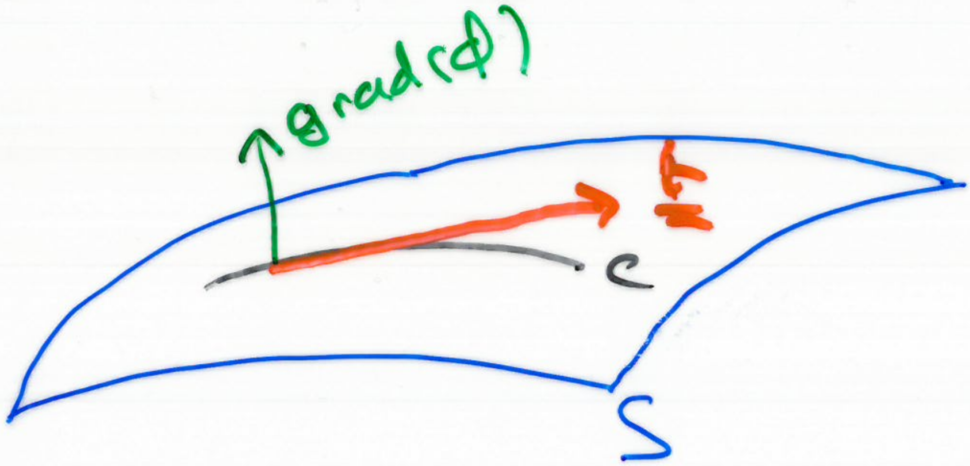
The equation

$$x^2 + y^2 + z^2 = 9$$

describes a sphere of radius 3 centred at the origin.

Let C be a curve on the surface S, parametrized as

$$C : \mathbb{R} \to S, \quad t \longmapsto (x(t), y(t), z(t)).$$



Note that

$$\phi(x(t), y(t), z(t)) = k$$

k the constant, whatever the curve C.

The chain rule gives

$$0 = \frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial t}$$

$$0 = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) \cdot \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right)$$

Now

$$\left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right) = \frac{\partial x}{\partial t}\underline{\underline{i}} + \frac{\partial y}{\partial t}\underline{\underline{j}} + \frac{\partial z}{\partial t}\underline{\underline{k}}$$

is a tangent $\underline{\underline{t}}$ to the curve C in the surface S.

Thus

$$\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) = \text{grad}\,\phi$$

is a vector, depending on (x, y, z) which is perpendicular to the surface S.