Differentiation of 1-forms

for 1-forms ω and ω' and for 0-forms $A, B, C, \dots$ in variables $x, y, z, \dots$

1. $d(\omega + \omega') = d\omega + d\omega'$
2. $dA = \dfrac{\partial A}{\partial x}dx + \dfrac{\partial A}{\partial y}dy + \dfrac{\partial A}{\partial z}dz + \dots$
3. $d(A\,dx + B\,dy + \dots)$
   $= (dA)\wedge dx + (dB)\wedge dy + \dots$
4. $dx \wedge dx = 0, \quad dy \wedge dy = 0, \dots$
5. $dx \wedge dy = -dy \wedge dx$
6. $(\omega + \omega')\wedge dx = \omega \wedge dx + \omega' \wedge dx$

<u>Example</u> Calculate $d\omega$ for

$$\omega = xy\,dz + yz\,dx + zx\,dy.$$

$$d\omega = d(xy\,dz) + d(yz\,dx) + d(zx\,dy)$$

$$= d(xy)\wedge dz + d(yz)\wedge dx + d(zx)\wedge dy$$

$$= (y\,dx + x\,dy)\wedge dz$$
$$+ (z\,dy + y\,dz)\wedge dx$$
$$+ (z\,dx + x\,dz)\wedge dy$$

$$= y\,dx\wedge dz + x\,dy\wedge dz$$
$$+ z\,dy\wedge dx + y\,dz\wedge dx$$
$$+ z\,dx\wedge dy + x\,dz\wedge dy$$

$$= \cancel{-y\,dz\wedge dx} + \cancel{x\,dy\wedge dz}$$
$$\cancel{-z\,dx\wedge dy} + \cancel{y\,dz\wedge dx}$$
$$+ \cancel{z\,dx\wedge dy} - \cancel{x\,dy\wedge dz}$$

$$= 0.$$

Last lecture we saw that rules 1-6 were sufficient to ensure

for $\omega = A\,dx + B\,dy$

we get

$$\boxed{d\omega = \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx \wedge dy \quad (*)}$$

To motivate rules 1-6 we'll explain why (*) is precisely what is needed for Stokes' formula to hold.

So suppose

$$\omega = A\,dx + B\,dy$$

where A, B are functions of x and y.

We want to define

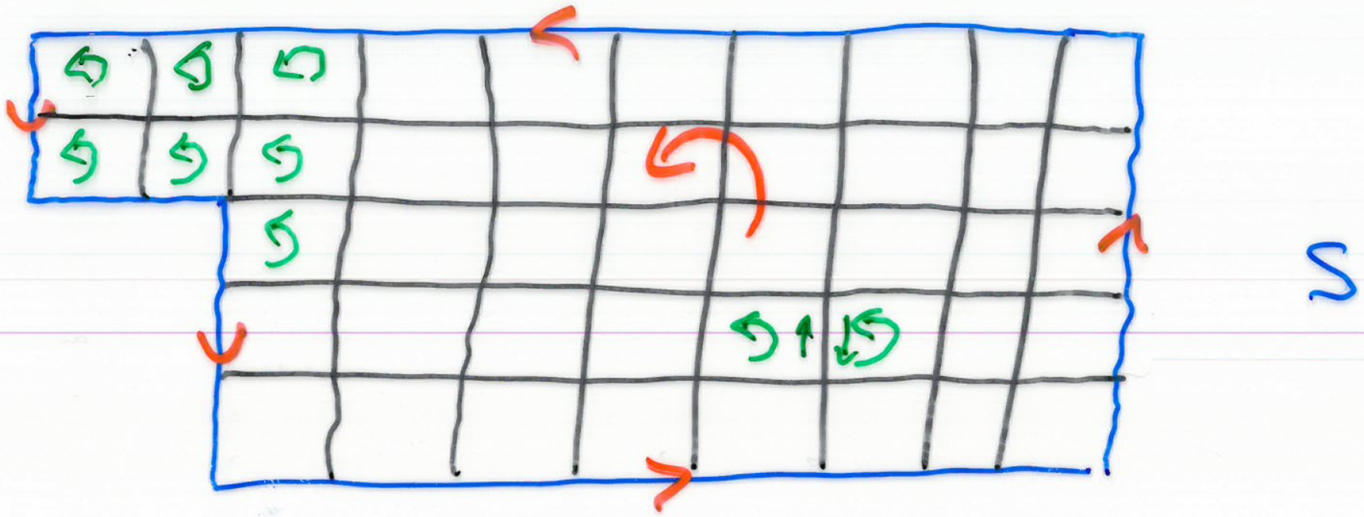
$$d\omega = C\,dx \wedge dy$$

where C is a function of x and y, such that

$$\boxed{\int_{\partial S} A\,dx + B\,dy = \int_S C\,dx \wedge dy}$$

what does C have to be?

For simplicity lets suppose that S is an oriented region in the xy-plane, with boundary ∂S oriented accordingly.



$$S = S_1 \cup S_2 \cup \cdots\cdots \cup S_n$$

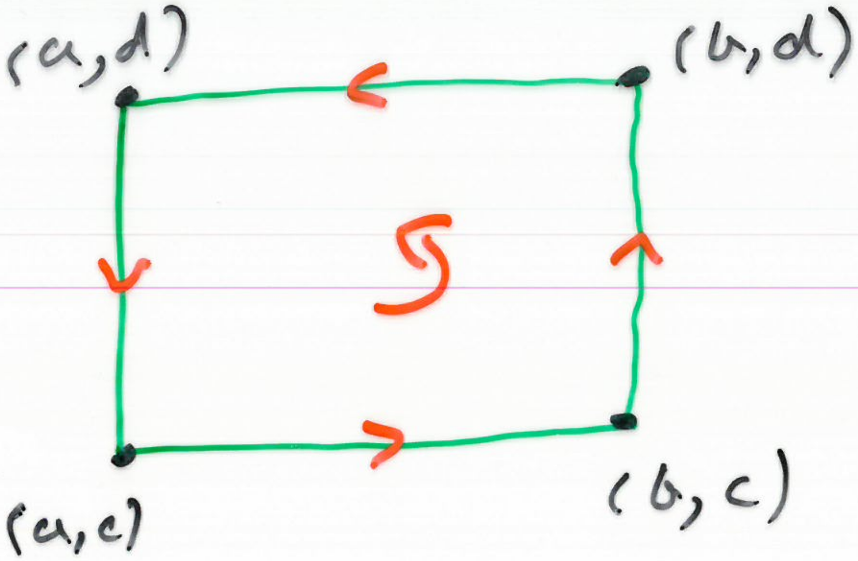
Note:

$$\int_{\partial S} A\,dx + B\,dy = \sum_{i=1}^{n} \int_{\partial S_i} A\,dx + B\,dy$$

So for each small S_i we just need

$$\boxed{\int_{\partial S_i} A\,dx + B\,dy = \int_{S_i} c\, dx \wedge dy \qquad (**)}$$

Suppose S_i is the square



$$a \le x \le b, \qquad c \le y \le d$$

Assume that a function C exists such that (**) holds.

We have

$$\int_{\partial S_i} A\,dx + B\,dy$$

$$= \int_a^b A(x,c)\,dx + \int_c^d B(b,y)\,dy$$

$$+ \int_b^a A(x,d)\,dx + \int_d^c B(a,y)\,dy$$

$$= \int_c^d \left(B(b,y) - B(a,y) \right) dy$$

$$- \int_a^b \left(A(x,d) - A(x,c) \right) dx$$

$$\underset{FTC}{=} \int_c^d \left(\int_a^b \frac{\partial B}{\partial x} \, dx \right) dy$$

$$- \int_a^b \left(\int_c^d \frac{\partial A}{\partial y} \, dy \right) dx$$

$$= \int_{S_i} \frac{\partial B}{\partial x} \, dx \wedge dy$$

$$- \int_{S_i} \frac{\partial A}{\partial y} \, dx \wedge dy$$

$$= \int_{S_i} \underbrace{\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right)}_{= C} dx \wedge dy$$

Thus we need

$$d\omega = c \, dx \wedge dy = \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx \wedge dy$$