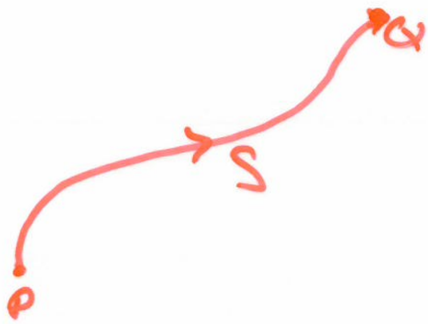
Proof of the fundamental Theorem of Calculus

We want to prove

$$\int_S d\omega = \int_{\partial S} \omega$$



where ω is a 0-form.

For simplicity let's consider the case of $n=2$ variables

Proof

Choose $\omega = F(x,y)$

Suppose $x = g(t)$, $y = h(t)$ is a parametrization of S as t varies from t_0 to t_1.

$$\int_S d\omega = \int_S F_x(x,y)\,dx + F_y(x,y)\,dy$$

$$= \int_{t_0}^{t_1} F_x(g(t),h(t))\,g'(t)\,dt + F_y(g(t),h(t))\,h'(t)\,dt$$

$$= \int_{t_0}^{t_1} \big(F_x(g(t),h(t))\,g'(t) + F_y(g(t),h(t))\,h'(t) \big)\,dt$$

chain rule

$$= \int_{t_0}^{t_1} \left(\frac{dF}{dt} \right) dt$$

$$= F(g(t_1),h(t_1)) - F(g(t_0),h(t_0))$$

FTC in one variable

$$= F(P) - F(Q)$$

$$=: \int_{\partial S} \omega \, .$$

Summary of 1-forms and an introduction to 2-forms

- A 1-form is an expression such as

$$\omega = A\,dx + B\,dy$$

that can be integrated over oriented curves.

- A 2-form is an expression such as

$$\omega = A\,dx\wedge dy + B\,dy\wedge dz + C\,dz\wedge dx$$

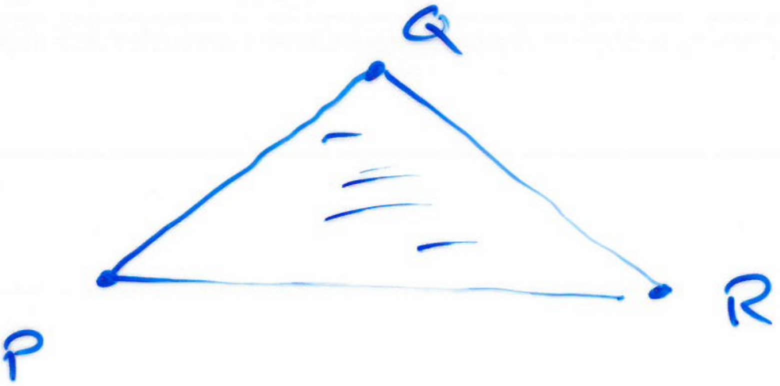
that can be "integrated" over 2-dimensional "oriented regions".

- integrals of 1-forms are just limits of sums of integrals of constant 1-forms over oriented straight line segments.

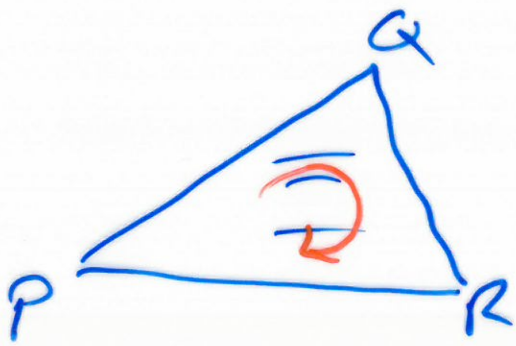
- integrals of 2-forms are just limits of sums of integrals of "constant 2-forms" over "oriented planar triangles".

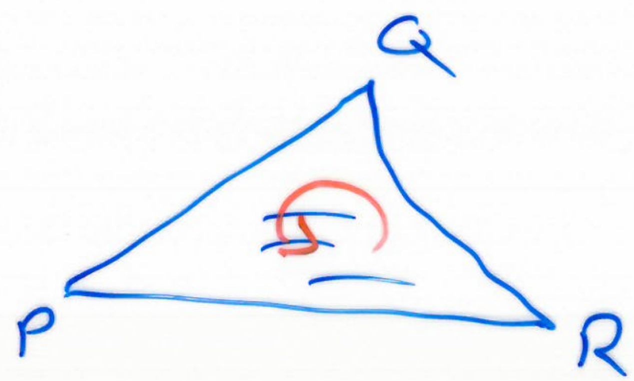
Oriented planar triangles

Three points in a plane determine a triangle



An orientation of a triangle is specified by an arrow





corresponding to one of two possible directions of rotation.

The positive side of an oriented triangle is the one for which the arrow denotes anti-clockwise rotation.

An orientation is just an ordering of the vertices of a triangle. The ordering RQP denotes the second triangle above. So too does PRQ. So too does QPR.