Sandwich Lemma

Suppose

$$f(x) \leq g(x) \leq h(x)$$

for all x near a (except possible for $x = a$). Suppose also

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = l$$

Then

$$\lim_{x \to a} g(x) = l$$

Example Evaluate

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right).$$

Sol^n

$$g(x) = x^2 \sin\left(\frac{1}{x}\right)$$

$$f(x) = -x^2$$

$$h(x) = x^2$$

For x near 0 and $x \neq 0$ we have

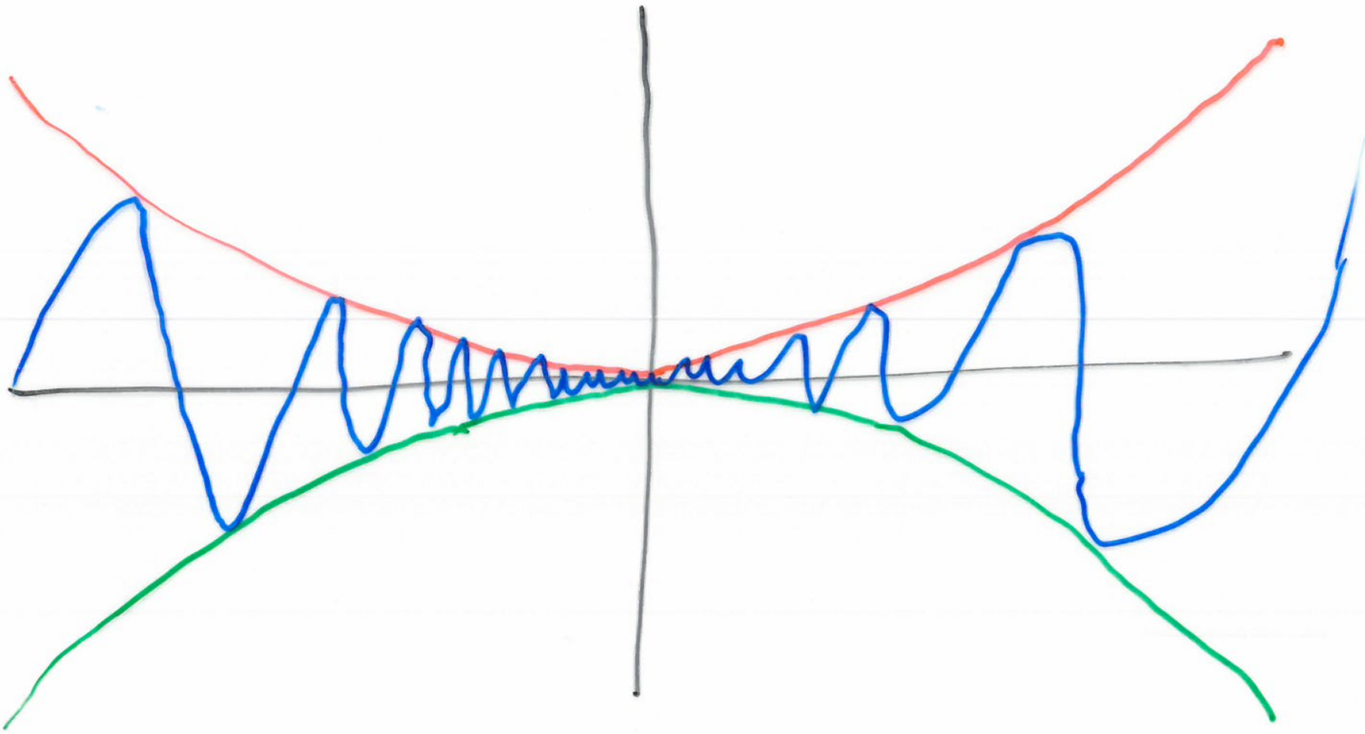
$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2 .$$

$$\lim_{x \to 0} -x^2 = 0$$

$$\lim_{x \to 0} x^2 = 0$$

The Sandwich Lemma tells us that

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0 .$$



Left-hand and right-hand limits

We write

$$\lim_{x \to a^-} f(x) = l$$

to mean that $f(x)$ is close to l for all x sufficiently close to a and strictly less than a.

<u>Example</u> Let $f(t)$ denote the car park charge in Dublin Airport for t hours of parking.

More precisely

$$f(t) = \begin{cases} 0 , & 0 \le t < 0.25 \\ 2 , & 0.25 \le t \le 1 \\ 4t , & 1 < t < \infty \end{cases}$$

$$\lim_{t \to 0.25^-} f(t) = 0$$

$$f(0.25) = 2 .$$

Example

Let $f(x) = \begin{cases} x^2+1 & \text{if } x < 3 \\ x+8 & \text{if } x \geq 3 \end{cases}$

Domain of $f = \mathbb{R}$ $\qquad f : \mathbb{R} \to \mathbb{R}$

Codomain of $f = \mathbb{R}$

$$\lim_{x \to 3-} f(x) = 10$$

$f(10) = 18$

Analogously

$$\lim_{x \to 3+} f(x) = 11$$

Proposition

$$\lim_{x \to a} f(x) = l$$

if and only if

$$\lim_{x \to a-} f(x) = l = \lim_{x \to a+} f(x).$$

if $\lim_{x \to a-} f(x) \neq \lim_{x \to a+} f(x)$

then $\lim_{x \to a} f(x)$ does not exist.

Example Let

$$f(x) = \begin{cases} x^2+1 & \text{if } x < 3 \\ x+c & \text{if } x \geq 3, \end{cases}$$

where c is a constant.

For what value of c does

$$\lim_{x \to 3} f(x)$$

exist?

Sol^n

$c = 7$ because then

$$\lim_{x \to 3-} f(x) = 10$$

and

$$\lim_{x \to 3+} = 10.$$

Since $\lim_{x \to 3-} f(x) = 10 = \lim_{x \to 3+} f(x)$

we have

$$\lim_{x \to 3} f(x) = 10.$$