Consider $g(x) = \dfrac{x^6 - 1}{x - 1}$.

Domain $= \mathbb{R} \setminus \{1\}$

$$g(0.9) = \frac{(0.9)^6 - 1}{0.9 - 1} = 4.685559\ldots$$

$$g(1.1) = \frac{(1.1)^6 - 1}{1.1 - 1} = 7.71561$$

$$g(0.99) = \frac{(0.99)^6 - 1}{0.99 - 1} = 5.8519\ldots$$

$$g(0.999) = \frac{(0.999)^6 - 1}{0.999 - 1} = 5.9850199\ldots$$

We write

$$\lim_{x \to 1} g(x) = 6$$

to mean that for all real numbers x "sufficiently close" to 1, but distinct from 1, the value of $g(x)$ "gets close" to 6.

Example Evaluate

$$\lim_{x \to 0} \frac{\sqrt{4+x} - 2}{x}$$

Sol^n

for $x \neq 0$ and close to 0

$$\frac{\sqrt{4+x} - 2}{x} \frac{(\sqrt{4+x} + 2)}{(\sqrt{4+x} + 2)}$$

$$= \frac{4+x-4}{x(\sqrt{4+x}+2)} = \frac{\cancel{x}}{\cancel{x}(\sqrt{4+x}+2)}$$

$$= \frac{1}{\sqrt{4+x}+2}$$

So $\displaystyle\lim_{x \to 0} \frac{\sqrt{4+x} - 2}{x} = \lim_{x \to 0} \frac{1}{\sqrt{4+x}+2} = \frac{1}{4}.$

Example Evaluate

$$\lim_{x \to 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4} \right).$$

Sol^n For $x \neq 2$ and x close to 2

$$\frac{1}{x-2} - \frac{4}{x^2-4}$$

$$= \frac{1}{x-2} - \frac{4}{(x+2)(x-2)}$$

$$= \frac{x+2-4}{(x+2)(x-2)}$$

$$= \frac{\cancel{x-2}}{(x+2)\cancel{(x-2)}}$$

$$= \frac{1}{x+2}.$$

So

$$\lim_{x \to 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4} \right) = \lim_{x \to 2} \frac{1}{x+2} = \frac{1}{4}.$$

Recall $|-3| = 3$

$|4| = 4$

Example Evaluate

$$\lim_{x \to 0} \frac{x}{|x-1| - |x+1|}$$

Sol^n For $x \neq 0$ and x close to 0

$$\frac{x}{|x-1| - |x+1|}$$

$$= \frac{x}{1-x - (x+1)}$$

$$= \frac{x}{-2x}$$

$$= -\frac{1}{2}.$$

So

$$\lim_{x \to 0} \frac{x}{|x-1| - |x+1|} = \lim_{x \to 0} \left(-\frac{1}{2}\right) = -\frac{1}{2}.$$

Correct definition of a limit:

We write

$$\lim_{x \to 1} g(x) = 6$$

to mean that

for any number $\varepsilon > 0$

there exists a number $\delta > 0$

such that

$$0 < |x-1| < \delta$$

implies

$$|g(x) - 6| < \varepsilon.$$