Last week:

$y(t)$ = world population at time t.

Malthusian Law

Rate of population growth is proportional to the population size.

$$\boxed{\frac{dy}{dt} = ky}$$

Model works well for small population size. $y = Ae^{kt}$

Model does not work well when t is large.

Today:

$$\boxed{\frac{dy}{dt} = ky - ly^2}$$

Logistic Model

where l is a tiny constant and so ly^2 is only significant when y is large, and y^2 is enormous.

Suppose $F(y)$ is an antiderivative of

$$f(y) = \frac{1}{ky - ly^2}.$$

Then:

$$\boxed{\frac{dy}{dt} = ky - ly^2} \quad (1)$$

$$\Longleftrightarrow$$

$$\boxed{\frac{1}{ky - ly^2} \frac{dy}{dt} = 1}$$

$$\Longleftrightarrow$$

$$\boxed{F(y) = t} \quad (2)$$

So we'd like to find an anti-derivative $F(y)$ of

$$\frac{1}{ky - ly^2}.$$

Suppose

$$f(y) = \frac{1}{ky - ly^2} = \frac{1}{y(k - ly)} = \frac{A}{y} + \frac{B}{k - ly}$$

where A, B are constants.

This would mean

$$\frac{1}{y(k-ly)} = \frac{A(k-ly)}{y(k-ly)} + \frac{By}{y(k-ly)}$$

So

$$1 = A(k-ly) + By$$

$$1 = Ak + (B - Al)y$$

Need

$$1 = Ak$$

$$0 = B - Al$$

So $A = \frac{1}{k}$, $B = \frac{l}{k}$.

In summary:

$$\boxed{f(y) = \frac{1}{ky - ly^2} = \frac{1}{ky} + \frac{l}{k^2 - kly}}$$

Thus an anti-derivative of $f(y)$ is

$$F(y) = \frac{1}{k}\ln(ky) - \frac{\cancel{l}}{k\cancel{l}}\ln(k^2 - kly)$$

$$F(y) = \frac{1}{k}\left(\ln(ky) - \ln(k^2 - kly)\right)$$

$$F(y) = \frac{1}{k}\ln\left(\frac{ky}{k^2 - kly}\right)$$

$$F(y) = \frac{1}{k}\ln\left(\frac{y}{k - ly}\right)$$

Equation (2) becomes

$$\frac{1}{k}\ln\left(\frac{y}{k-ly}\right) = t$$

$\Leftrightarrow$

$$kt = \ln\left(\frac{y}{k-ly}\right)$$

$\Leftrightarrow$

$$e^{kt} = \frac{y}{k-ly}$$

$\iff$

$$(k - ly)e^{kt} = y$$

$\iff$

$$ke^{kt} = y(1 + le^{kt})$$

$$\boxed{y = \frac{ke^{kt}}{1 + le^{kt}}}$$

So $y \to \dfrac{ke^{kt}}{le^{kt}} = \dfrac{k}{l}$ as $t \to \infty$.

Conclusion: The logistic model implies that the population of the world will tend to some constant k/l.

we can estimate k and l from our knowledge of past world population sizes.

An estimate, using 1950, 1960 and 1970 populations is

$$y(t) \to \frac{k}{l} \simeq 9.86 \text{ billion}$$