Antiderivatives

A function $F(x)$ is called an **antiderivative** of $f(x)$ if

$$\frac{d}{dx}F(x) = f(x).$$

Example An antiderivative of $f(x) = \frac{1}{x+1}$ is

$$F(x) = \ln(x+1)$$

Example An antiderivative of $f(x) = \frac{1}{3} x e^{x^2}$ is

$$F(x) = \frac{1}{6} e^{x^2}.$$

check:

$$\frac{d}{dx}\left(\frac{1}{6}e^{x^2}\right) = \frac{1}{6} e^{x^2} 2x = \frac{1}{3} x e^{x^2}$$

Lemma Any two anti-derivatives of $f(x)$ differ by a constant.

Proof If $F(x)$ and $G(x)$ are both anti-derivatives of $f(x)$ then

$$\frac{d}{dx}\left(F(x) - G(x)\right) = \frac{d}{dx}F(x) - \frac{d}{dx}G(x)$$

$$= f(x) - f(x)$$

$$= 0.$$

Hence $F(x) - G(x) = C$ a constant. $\boxtimes$

Notation If $F(x)$ is an anti-derivative of $f(x)$ then we write

$$\int f(x)\,dx = F(x) + C$$

Example

$$\int \cos(x)\,dx = \sin(x) + c$$

Example observations suggest that the rate of growth of zebra mussels is exponential in time.

Suppose

$$\frac{dy}{dt} = e^{2t}$$

where $y(t)$ is the number of mussels at time t (days).

If there are 150 mussels at time $t=0$, how many mussels will there be after ten days.

Sol^n If $\frac{dy}{dt} = e^{2t}$ then

$$\boxed{y(t) = \frac{1}{2}e^{2t} + c}.$$

$y(0) = 150$

Need to find $y(10)$.

$y(0) = 150 = \frac{1}{2}e^{0} + C = \frac{1}{2} + C.$

$c = 149.5$

So $\boxed{y(t) = \frac{1}{2}e^{2t} + 149.5}$

At $t = 10$ days the population of mussels is

$y(10) = \frac{1}{2}e^{20} + 149.5 \simeq 2.4 \times 10^{8}$ mussels.

Basic rules of anti-derivatives

a) if $f(x) = x^n$ then

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$$

(for $n \neq -1$)

b) if $f(x) = \sin(x)$ then

$$\int \sin(x) \, dx = -\cos(x) + C.$$

c) if $f(x) = \cos(x)$ then

$$\int \cos(x) \, dx = \sin(x) + C$$

d) $$\int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx$$

<u>Example</u> The world's population was 2560 millions in 1950, and 3040 millions in 1960.

Assume that the growth rate of the population is proportional to the size of the world's population.

Thomas Malthus

a) Estimate the population in 1990

b) " " " " 2040

<u>Solⁿ</u>

Let $y(t)$ = population of the world at time t, in millions of people.

$$\boxed{\frac{dy}{dt} = k y}$$

where k is a constant.

$$\boxed{y = Ae^{kt}}$$

$t=0$ in year 1950

$$y(0) = 2560 = Ae^{k.0} = A$$

$$\boxed{A = 2560}$$

so $$\boxed{y = 2560\, e^{kt}}$$

$t=10$ in 1960.

$$y(10) = 3040 = 2560\, e^{10k}$$

$$\frac{3040}{2560} = e^{10k}$$

$$\ln\left(\frac{3040}{2560}\right) = 10k$$

$$k = \frac{1}{10}\ln\left(\frac{3040}{2560}\right) \simeq 0.017185$$

So

$$\boxed{y(t) = 2560\, e^{0.01785\, t}}$$

In 1990 $t = 40$, and the population should have been

$$y(40) = 2560\, e^{0.01785 \times 40}$$

$$\simeq 5090 \text{ millions}$$

$$\boxed{\text{Actual population in 1990 was } 5278 \text{ millions}}$$

In 2040, $t = 90$ and the world's population should be

$$y(90) = 2560\, e^{0.01785 \times 90}$$

$$\simeq 12021 \text{ millions.}$$