Topic 3

Differential Equations:

An equation involving a derivative, such as

$$\frac{dy}{dt} = ky \qquad (*)$$

where k is a constant and $y = f(t)$ is a function of t.

Are there any solutions to the equation $(*)$?

Consider

$$y = e^{kt}$$

$$\frac{dy}{dt} = k e^{kt} = ky$$

So $y = e^{kt}$ is one solution of $(*)$.

Another solution is

$$y = 12e^{kt}$$

$$\frac{dy}{dt} = \underline{12e^{kt}} \cdot k = ky$$

In fact the function

$$y = Ae^{kt}$$

is a solution to (*) for any constant A.

Question: Are there any other solutions to (*)?

Suppose $y = y(t)$ and $z = z(t)$ are both solutions to the diff. eqn. (*).

Then

$$\frac{d}{dt}\left(\frac{y}{z}\right) = \frac{z'y - y'z}{z^2}$$

$$= \frac{kzy - kyz}{z^2}$$

$$= 0.$$

So $\frac{y}{z}$ must be a constant, say

$$\frac{y}{z} = A,$$

or $y = Az$.

Conclusion:

$\boxed{\begin{array}{l}\text{The only solutions to the} \\ \text{diff. eq}^{n} \\ \qquad \dfrac{dy}{dt} = ky \qquad\qquad (*) \\ \text{are the functions} \\ \qquad y = Ae^{kt} \\ \text{with } A \text{ any constant.}\end{array}}$

<u>Problem</u> A cup of coffee in a room at 20°c cools from 80°c to 50°c in five minutes. How long will it take to cool to 60°c ?

Sol[4]

$\boxed{\text{Newton: A hot object cools at a rate proportional to the excess of its temperature above room temperature.}}$

t = time in minutes

$y(t)$ = temperature of coffee at time t.

$y(0) = 80$

$y(5) = 50$

Required to find value of t such that $y(t) = 40$.

Newton:

$$\frac{dy}{dt} = k(y-20)$$

Consider $z = y - 20$

$z(0) = 60$

$z(5) = 30$

$$\frac{dz}{dt} = \frac{d}{dt}(y-20) = \frac{d}{dt}y = k(y-20) = kz$$

so

$$\frac{dz}{dt} = kz \quad . \qquad (*)$$

Since z satisfies eqn $(*)$ we know that

$$z = Ae^{kt}$$

with A, k constants.

$z(0) = 60$

$$60 = Ae^{k.0} = A .$$

$\boxed{A = 60}$

so $\boxed{z = 60 e^{kt}}$.

$z(5) = 30$

$$30 = 60 e^{5k}$$

$$\boxed{\frac{1}{2} = e^{5k}}$$

Required to find the value of t such that $z(t) = 20$, or $y(t) = 40$.

$$20 = z(t) = 60\,e^{kt}$$

$$\frac{1}{3} = e^{kt}$$

$$\frac{1}{3} = \left(e^{5k}\right)^{\frac{t}{5}}$$

$$\frac{1}{3} = \left(\frac{1}{2}\right)^{\frac{t}{5}}$$

$$\ln\left(\frac{1}{3}\right) = \ln\left(\left(\frac{1}{2}\right)^{\frac{t}{5}}\right)$$

$$\ln\left(\frac{1}{3}\right) = \frac{t}{5}\ln\left(\frac{1}{2}\right)$$

$$t = \frac{5\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{1}{2}\right)} \simeq 7.92 \text{ minutes}$$