Some important terminology

- $f(x)$ is continuous at $x = c$ if $f(c)$ is defined and

$$\lim_{x \to c} f(x) = f(c).$$

- $f(x)$ is differentiable at $x = c$ if $f(c)$ is defined and the limit

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists.

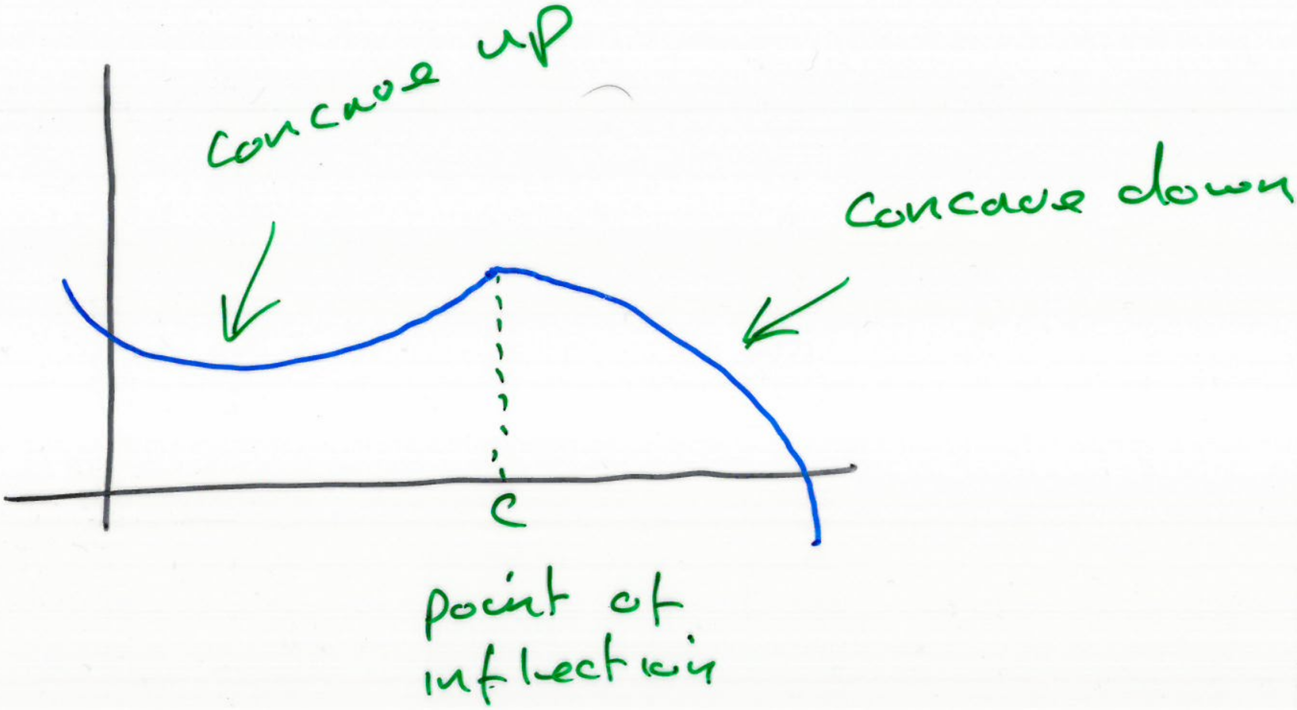
- $x = c$ is a critical point of $f(x)$ if either $f'(c)$ does not exist or $f'(c) = 0$.

- $f(x)$ is increasing on (a, b) if $f'(x) \geq 0$ for $x \in (a, b)$.

- $f(x)$ is concave up on (a, b) if $f''(x) \geq 0$ for $x \in (a, b)$.

(informally: "concave up" means accelerating)

- $x = c$ is a **point of inflection** of $f(x)$ if the concavity changes.



Theorem If $f(x)$ is differentiable at $x = c$ then $f(x)$ is continuous at $x = c$.

Proof

$f(x)$ is differentiable at $x = c$

$\Longleftrightarrow$

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = l \quad \text{exists}$$

$\Longrightarrow$

$$\left(\lim_{h \to 0} h\right)\left(\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}\right) = \left(\lim_{h \to 0} h\right) l$$

$\Longrightarrow$

$$\lim_{h \to 0} \left(\cancel{h} \cdot \frac{f(c+h) - f(c)}{\cancel{h}} \right) = 0$$

$\Longrightarrow$

$$\lim_{h \to 0} f(c+h) - \lim_{h \to 0} f(c) = 0$$

$\Longrightarrow$

$$\lim_{h \to 0} f(c+h) = f(c)$$

$\Longrightarrow$

$$\lim_{x \to c} f(x) = f(c)$$

$\Rightarrow$

$f(x)$ is continuous at $x = c$.

<u>Example</u> Find all possible values of a, b such that

$$f(x) = \begin{cases} x^2 + x + 1, & x \geq 1 \\ ax + b, & x < 1 \end{cases}$$

is differentiable at all points.

<u>Sol^n</u> $f(x)$ is "clearly" differentiable at all points $x \neq 1$,

Need to choose a, b such that $f(x)$ is differentiable at $x = 1$.

In particular, $f(x)$ must be continuous at $x = 1$, i.e.

$$\lim_{x \to 1} f(x) = f(1)$$

$$\iff$$

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = f(1).$$

$$\iff$$

$$\boxed{a + b = 3}$$

So choose $a + b = 3$.

Now

$f'(1)$ exists

$$\iff$$

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \quad \text{exists}$$

$$\iff$$

$$\lim_{h \to 0-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0+} \frac{f(1+h) - f(1)}{h}$$

$$\iff$$

$$q'(1) = p'(1)$$

where $q(x) = ax + b$ where $p(x) = x^2 + x + 1$

$\Longleftrightarrow$

$$\boxed{a = 3}$$

and

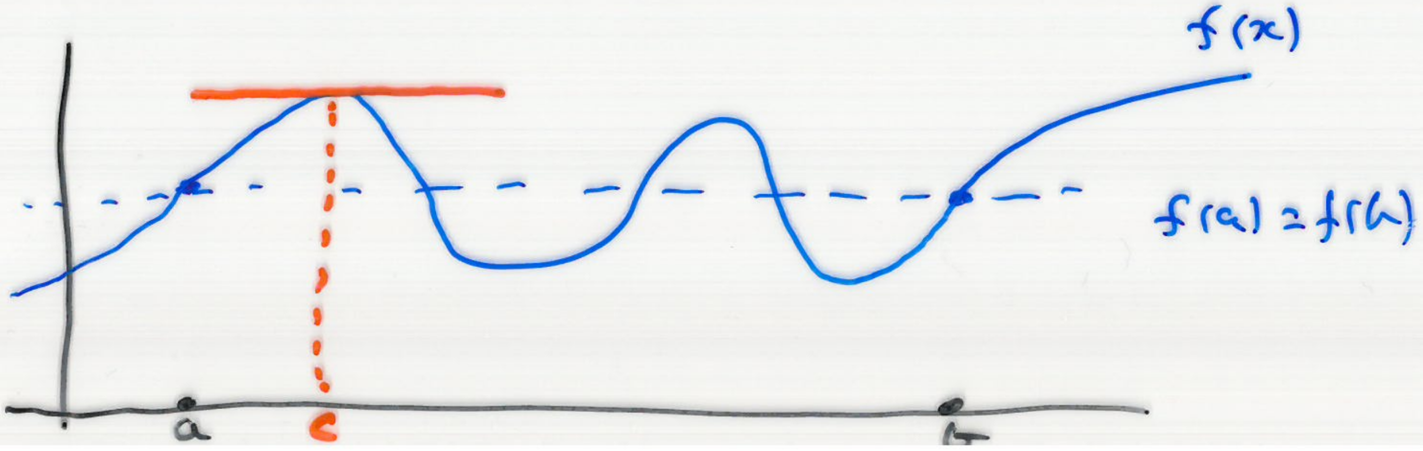
$$\boxed{b = 0}$$

Rolle's Theorem

Suppose $f(x)$

- is continuous on $[a, b]$
- differentiable on (a, b)
- and $f(a) = f(b)$.

Then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.



Exercise Prove that there is exactly one solution to

$$x^3 + x + 1 = 0.$$

Solⁿ