Frogs

Problem: The population of frogs on an island is infected with a disease. Each day 20% of the healthy frogs become ill, and 30% of the ill frogs become healthy.

There are 500 frogs on the island, of which 100 are initially infected.

Determine the number of infected frogs after 1, 2, 3, ... days, and investigate what happens in the long term.

x_n = number of healthy frogs on day n

y_n = " " ill " " " "

$x_0 = 400$

$y_0 = 100$

$x_n = 0.8x_{n-1} + 0.3y_{n-1}$

$y_n = 0.2x_{n-1} + 0.7y_{n-1}$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \underbrace{\begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}}_{A} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \begin{pmatrix} 400 \\ 100 \end{pmatrix} = \begin{pmatrix} 350 \\ 150 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \begin{pmatrix} 350 \\ 150 \end{pmatrix} = \begin{pmatrix} 325 \\ 175 \end{pmatrix}$$

$\vdots$

From last week:

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

if T is a 2×2 invertible matrix whose columns are eigenvectors of A with corresponding eigenvalues λ_1, λ_2.

Then

$$A = T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} T^{-1},$$

$$A^n = T\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cancel{T^{-1}} \cancel{T} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cancel{T^{-1}} \cdots \cancel{T} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} T^{-1}$$

$$\boxed{A^n = T \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} T^{-1}} \qquad (*)$$

from (*)

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = T \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} T^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

or

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \overbrace{T \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2^n} \end{pmatrix} T^{-1}}^{A^n} \begin{pmatrix} 400 \\ 100 \end{pmatrix} ,$$

for large n

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A \; \underbrace{A^n \begin{pmatrix} 400 \\ 100 \end{pmatrix}} = \underbrace{A^n \begin{pmatrix} 400 \\ 100 \end{pmatrix}}$$

eigenvector
of A
corresponding
to eigenvalue
$\lambda = 1$.

Let's find eigenvalues / vectors of

$$A = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}.$$

$$\det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} 0.8-\lambda & 0.3 \\ 0.2 & 0.7-\lambda \end{pmatrix} = 0$$

$$(0.8-\lambda)(0.7-\lambda) - (0.2)(0.3) = 0$$

$$\lambda^2 - 1.5\lambda + 0.56 - 0.06 = 0$$

$$\lambda^2 - 1.5\lambda + 0.5 = 0$$

$$(\lambda - 1)(\lambda - \tfrac{1}{2}) = 0$$

So the eigenvalues of A are

$\lambda_1 = 1$, $\lambda_2 = \frac{1}{2}$.

Let's find the eigenvector.

$$(A - \lambda_1 I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\lambda_1 = 1$

$$\begin{pmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{pmatrix}\begin{pmatrix} 300 \\ 200 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Conclusion

In the long run there will be 300 healthy frogs and 200 ill frogs each day.