suppose

$F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3, \ldots$

where

$$F_n = F_{n-1} + F_{n-2}.$$

Aim: find an explicit formula for F_n in terms of n but not involving $F_{n-1}, F_{n-2}, \ldots$

Theorem If a 2×2 matrix A has eigenvalues λ_1, λ_2 with corresponding eigenvectors v_1, v_2, and if the matrix

$$T = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$$

is invertible, then

$$T^{-1} A T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Proof

$$T^{-1} A T = T^{-1} A \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$$

$$= T^{-1} \begin{pmatrix} | & | \\ \lambda_1 v_1 & \lambda_2 v_2 \\ | & | \end{pmatrix}$$

$$= T^{-1} \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$= T^{-1} T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

QED

Example

Consider

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

We've seen that the eigenvalues of A are

$$\lambda_1 = \phi = \frac{1+\sqrt{5}}{2}$$

$$\lambda_2 = \bar{\phi} \qquad \frac{1-\sqrt{5}}{2}$$

Note: $\phi \bar{\phi} = -1$.

Let's find corresponding eigenvectors.

Need to solve

$$(A - \lambda I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for $\lambda = \phi, \bar{\phi}$.

Consider $\lambda = \phi$.

$$\begin{pmatrix} 1-\phi & 1 \\ 1 & -\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-\phi & 1 \\ 1 & -\phi \end{pmatrix} \underbrace{\begin{pmatrix} \phi \\ 1 \end{pmatrix}}_{\text{eigenvector for } \lambda = \phi} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Consider $\lambda = \bar{\phi}$

$$\begin{pmatrix} 1-\bar{\phi} & 1 \\ 1 & -\bar{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} 1 \\ -\phi \end{pmatrix}}_{\text{eigenvector for } \lambda = \bar{\phi}.} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Set $T = \begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix}$

By the above theorem

$$\boxed{T^{-1} A T = \begin{pmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{pmatrix}}$$

Recall from last lecture

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = A^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$

where $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

we have

$$\boxed{A = T \underbrace{\begin{pmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{pmatrix}}_{D} T^{-1}}$$

$$A^n = (TDT^{-1})^n$$

$$= (TD\cancel{T^{-1}})(\cancel{T}D\cancel{T^{-1}})(\cancel{T}D\cancel{T^{-1}}) \cdots\cdots (\cancel{T}DT^{-1})$$

$$= T\ D^n\ T^{-1}$$

$$= T \begin{pmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{pmatrix}^n T^{-1}$$

$$= T \begin{pmatrix} \phi^n & 0 \\ 0 & \bar{\phi}^n \end{pmatrix} T^{-1}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = A^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = T\, D^{n-1}\, T^{-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix} \begin{pmatrix} \phi^{n-1} & 0 \\ 0 & \bar{\phi}^{n-1} \end{pmatrix} \begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Exercise

$$F_n = \frac{1}{\sqrt{5}}\phi^n - \frac{1}{\sqrt{5}}\bar{\phi}^n$$

$$\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right) = \frac{1-5}{4} = -1$$

$$(1-\phi)\phi + 1 = \phi - \phi^2 + 1$$

$$= \frac{1+\sqrt{5}}{2} - \frac{1+2\sqrt{5}+5}{4} + 1$$

$$= \frac{1+\sqrt{5}}{2} - \frac{3+\sqrt{5}}{2} + 1$$

$$= \frac{1+\sqrt{5}}{2} - \frac{2}{2} - \frac{1+\sqrt{5}}{2} + 1$$

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$A^2 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$$

$$A^3 = A^2 A = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^3 & 0 \\ 0 & b^3 \end{pmatrix}$$

$$A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$$

<u>Rough work</u>

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^n \neq \begin{pmatrix} a^n & b^n \\ c^n & d^n \end{pmatrix} \quad \text{WRONG}}$$