Calculating eigenvalues & eigenvectors

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix of real numbers.

A non-zero vector $v = \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is said to be an eigenvector of A if

$$Av = \lambda v$$

for some number λ. We say that λ is an eigenvalue of A corresponding to v.

To calculate eigenvectors/values we need the following result.

<u>Proposition</u> Let A be a 2×2 matrix of real numbers, and let $v = \begin{pmatrix} x \\ y \end{pmatrix}$ be some non-zero vector, and suppose that

$$A v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (*)$$

Then $\det(A) = 0$.

<u>Proof</u> If A^{-1} existed then from $(*)$ we get

$$A^{-1} A v = A^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and so

$$v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We can conclude, under the hypothesis of the proposition, that A^{-1} does <u>not</u> exist.

Recall the formula

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) .$$

Since A^{-1} does not exist, and since $\operatorname{adj}(A)$ always exists, we get that

$\det(A) = 0$. QED

How can we find the eigenvalues & eigenvectors of

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} ?$$

Suppose $v = \begin{pmatrix} x \\ y \end{pmatrix}$ is some eigen-vector of A. Then

$$Av = \lambda v$$

for some number λ.

$$Av = \lambda v$$

$$Av - \lambda v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$Av - \lambda I v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(A - \lambda I) v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so

$$\det(A - \lambda I) = 0$$

by the proposition above and the fact that eigenvectors are non-zero.

So

$$\det\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = 0$$

and thus

$$\det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = 0$$

$$(2-\lambda)(2-\lambda) - 1.1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 3)(\lambda - 1) = 0$$

Hence the eigenvalues of A

are $\lambda = 1$, $\lambda = 3$.

now let's find the eigenvectors.

<u>Case $\lambda = 1$</u>

Need $Av = \lambda v$

or $Av = v$

or

$$Av - v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(A - I)v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So for instance $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of A corresponding to $\lambda = 1$.

Case $\lambda = 3$

Again, we need

$$(A - \lambda I) v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence

$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector

of A corresponding to $\lambda = 3$.

Rabbits

- One newly born male rabbit and one newly born female rabbit are placed in a field.
- Rabbits can mate at the age of one month, and one month later the female produces one male/ pair pair of kittens.
- Rabbits never die.

– How fast does the rabbit population grow?

- How many rabbits will there be after 100 months?

	month	number of pairs
MF	0	1
MF	1	1
MF MF	2	2
MF MF MF	3	3
MF MF MF MF MF	4	5
	5	8
	6	13
	7	21

Let F_n = number of pairs of rabbits after n months.

$$\boxed{F_{n+2} = F_{n+1} + F_n}$$

Q? What value is F_{100} ?