Last week:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det(A) = ad - bc$$

We'll also write $|A| = \det(A)$

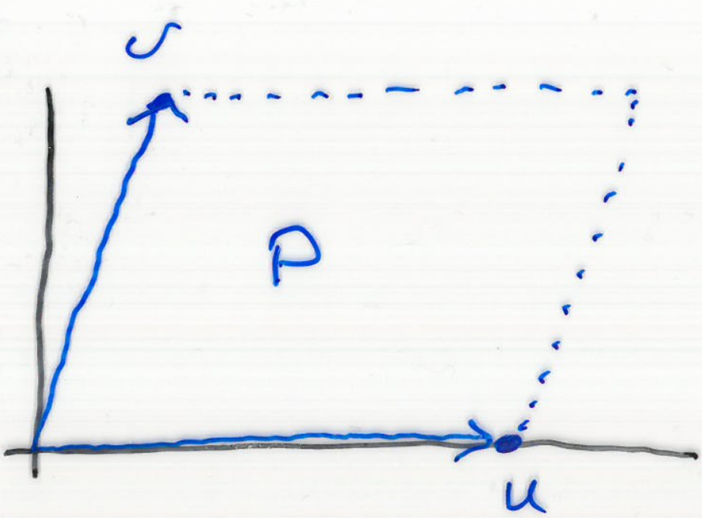
Example

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 5 \end{pmatrix}$$

$$\det(A) = |A| = 3.5 - 1.0 = 15$$

Consider

$$u = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$



Area of P $= 3 \times 5$

$= 15$

Note: in this example

$\det(A)$ = area of P.

The calculation of area of P was easy because u is on the x-axis,

Example

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -1 \\ 3 & -4 \end{pmatrix}$$

$$\det(A) = 1.4 - 3.2 = -2$$

$$\det(B) = 2(-4) - (3)(-1) = -5$$

$$\det(AB) = \det \begin{pmatrix} 8 & -9 \\ 18 & -19 \end{pmatrix}$$

$$= 8(-19) - 18(-9)$$

$$= 10.$$

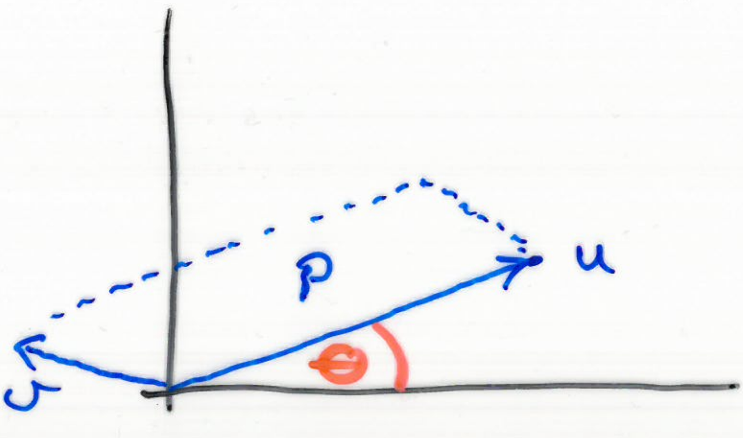
note: $\det(A)\det(B) = \det(AB)$.

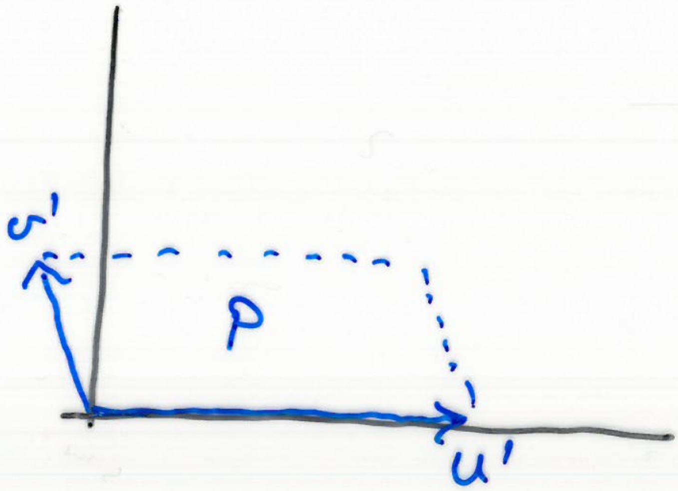
It's easy to prove:

Theorem For any 2x2 matrices A, B we have

$$|AB| = |A| \times |B| .$$

Towards a proof that areas of parallelograms are captured by determinants.





$$u = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} u'$$

$$v = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} v'$$

So

$$\det\begin{pmatrix} | & | \\ u & v \\ | & | \end{pmatrix}$$

$$= \det\left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} | & | \\ u' & v' \\ | & | \end{pmatrix} \right)$$

$$= \det\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \det\begin{pmatrix} | & | \\ u' & v' \\ | & | \end{pmatrix}$$

$$= \underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} (\pm \text{ area of } P)$$

$$= \pm \text{ area of } P.$$

Eigenvalue & Eigenvector

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix of real numbers.

Definition A non-zero vector

$$v = \begin{pmatrix} x \\ y \end{pmatrix}$$

is an eigenvector for A if there exist some real number λ such that

$$A v = \lambda v.$$

we call λ the eigenvalue of A corresponding to the vector v.

Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Consider

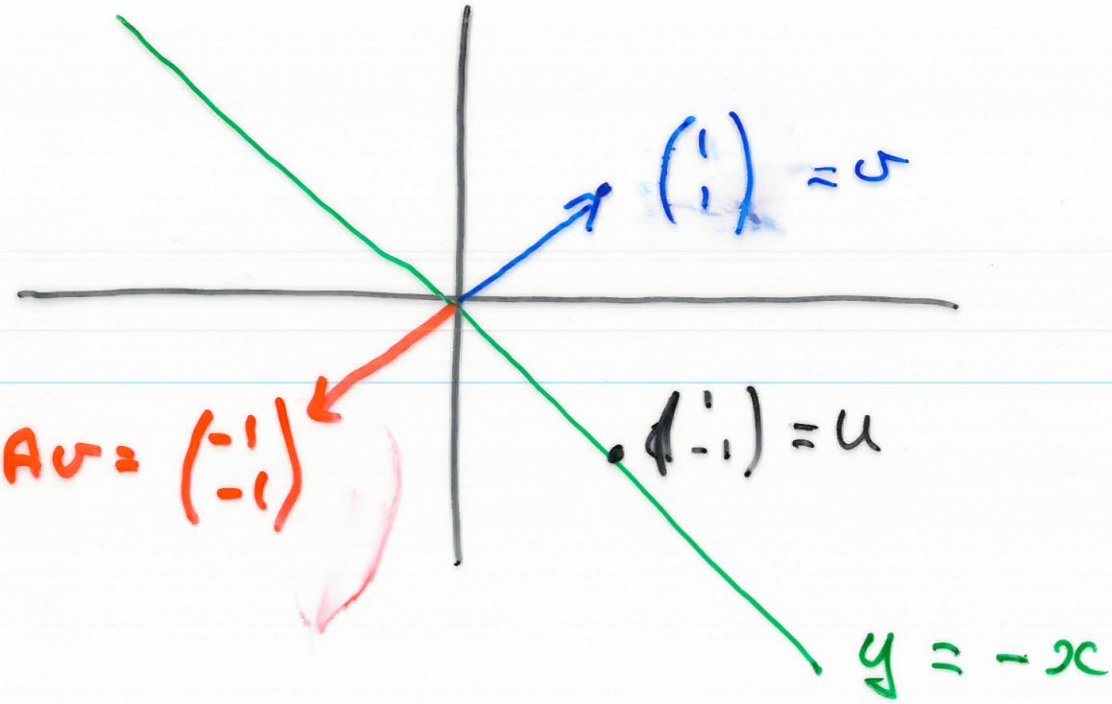
$$v = \begin{pmatrix} 4 \\ 4 \end{pmatrix}.$$

Then

$$\underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 4 \\ 4 \end{pmatrix}}_{v} = \begin{pmatrix} 12 \\ 12 \end{pmatrix} = 3 \underbrace{\begin{pmatrix} 4 \\ 4 \end{pmatrix}}_{v}$$

Thus $v = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ is an eigenvector of A with eigenvalue $\lambda = 3$.

Example Let A be the matrix of reflection in the line $y = -x$.



A represents reflection in line $y = -x$.

So $Av = -v$ for $v = \begin{pmatrix}1\\1\end{pmatrix}$.

Hence $v = \begin{pmatrix}1\\1\end{pmatrix}$ is an eigenvector for A with eigenvalue $\lambda = -1$.

note $\begin{pmatrix}2\\2\end{pmatrix}$ is also an eigenvector of A with eigenvalue $\lambda = -1$.

Also $Au = u$ for $u = \begin{pmatrix}1\\-1\end{pmatrix}$. Hence $u = \begin{pmatrix}1\\-1\end{pmatrix}$ is an eigenvector for A with eigenvalue $\lambda = 1$.

Example Give me a matrix A that has no eigenvectors.

Answer: Let A be the matrix of rotation about the origin through an angle θ, with $\theta = 0, \pi$.

Then "clearly" A has no eigenvectors.