Last week: we checked that

$T : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (x+2y, 3x+4y)$

was linear, i.e.

- $T(P+Q) = T(P) + T(Q)$
- $T(\lambda P) = \lambda T(P)$ for $\lambda \in \mathbb{R}$.

Note:

$$T(x, y) = (x+2y, 3x+4y)$$

can be represented as matrix multiplication

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 3x+4y \end{pmatrix}$$

We say that the matrix

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

represents T.

Theorem Any linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ can be represented by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Proof

well $T(1,0) = (a, b)$ say,

and $T(0,1) = (c, d)$ say.

$$T(x,y) = T\big(x(1,0) + y(0,1)\big)$$

by linearity
$$\left\{\begin{aligned} &= T(x(1,0)) + T(y(0,1)) \\ &= xT(1,0) + yT(0,1) \end{aligned}\right.$$

$$= x(a, b) + y(c, d)$$

$$= (xa, xb) + (yc, yd)$$

$$= (xa + yc, xb + yd)$$

now

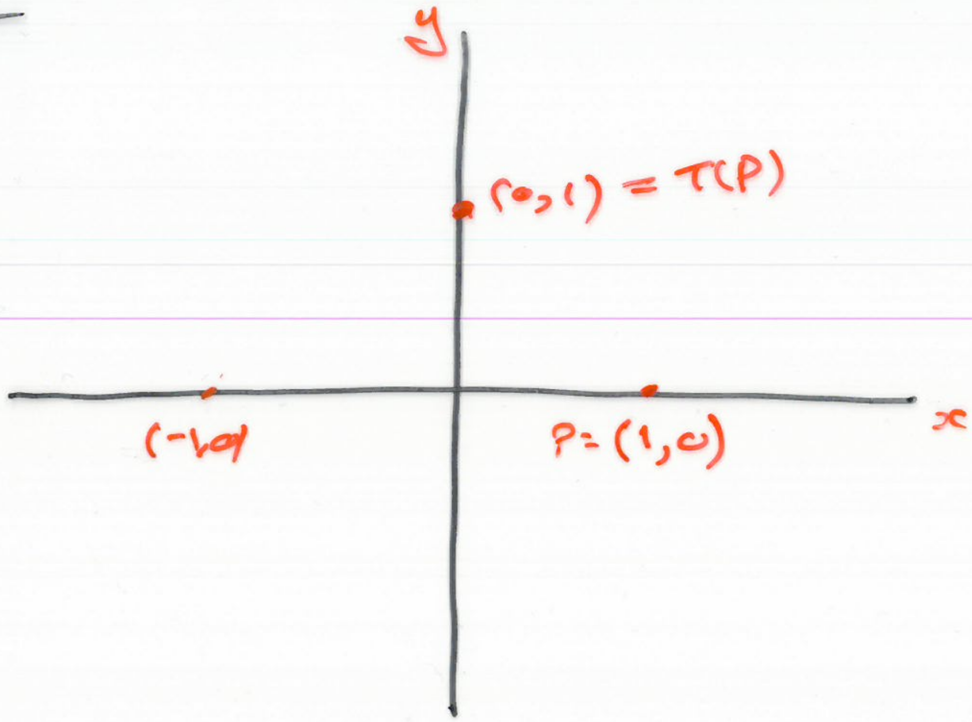
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + cy \\ bx + dy \end{pmatrix}$$

Facts:

- Any reflection in a line through the origin is linear.
- Any rotation of the plane about the origin is linear.
- Any composite of linear transformations is linear.

Example: Find the matrix representing a reflection in the y-axis, followed by a clockwise rotation of $\frac{5\pi}{2}$ rads about the origin.

Soln



$$T(1,0) = (\overset{a}{0}, \overset{b}{1})$$

$$T(0,1) = (\underset{c}{1}, \underset{d}{0})$$

So T is represented by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Theorem Let

$S: \mathbb{R}^2 \to \mathbb{R}^2$ and $T: \mathbb{R}^2 \to \mathbb{R}^2$

be linear transformations

represented by matrices

A and B.

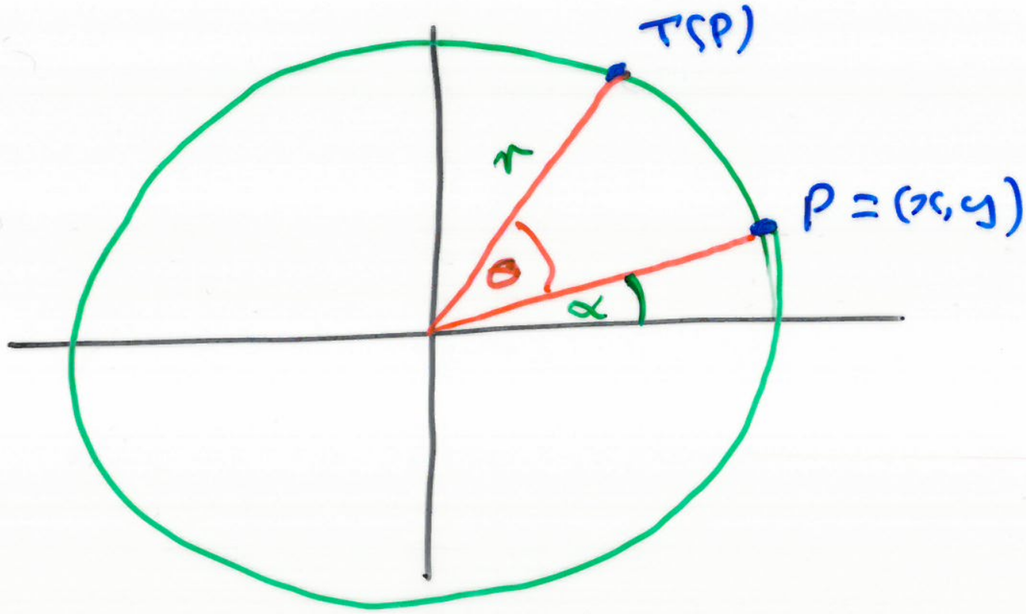
Then the linear transformation

$S \circ T : \mathbb{R}^2 \to \mathbb{R}^2, \quad v \mapsto S(T(v))$

is represented by the matrix

AB.

Consider an anticlockwise rotation of the plane about the origin through an angle θ. What matrix represents this transformation?



If $P = (x, y) = (r\cos\alpha, r\sin\alpha)$

then

$$T(P) = (r\cos(\theta+\alpha), r\sin(\theta+\alpha))$$

$$= r(\cos(\theta+\alpha), \sin(\theta+\alpha))$$

$$= r(\cos\alpha\cos\theta - \sin\alpha\sin\theta,\ \sin\alpha\cos\theta + \sin\theta\cos\alpha)$$

$$= (x\cos\theta - y\sin\theta,\ y\cos\theta + x\sin\theta)$$

So

$$T(P) = \begin{pmatrix} x\cos\theta - y\sin\theta \\ \\ y\cos\theta + x\sin\theta \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \\ \sin\theta & \cos\theta \end{pmatrix}}_{\text{matrix of anticlock rotation about origin through an angle } \theta.} \begin{pmatrix} x \\ \\ y \end{pmatrix}$$