Some Notes for MA342

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Notation

- "$\Rightarrow$" means "implies that".
- "$\Leftrightarrow$" means "if and only if".
- $\mathbb{N}$ denotes the set of natural numbers.
- $\mathbb{Z}$ denotes the set of integers.
- $\mathbb{Q}$ denotes the set of rational numbers.
- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{C}$ denotes the set of complex numbers.
- $\mathbb{R}^+$ denotes the set of strictly positive real numbers.

Contents

CHAPTER 0

Some Motivation for Topology

I will mention a few problems that have 'topological connections'. The idea is to give an impression of the breadth of application of the subject rather than to explain the detail of each application. Topology is the underlying mathematical theory that connects all of these problems. Our course will not necessarily explain how to solve these problems, but we will see how to put these problems into the appropriate mathematical context.

1. Knots

Can we classify different types of knots. What does that mean? Recently biologists have become interested in study knots that arise in protein structures.

2. Networks and Graphs

Suppose that Anne, Barry and Cathy work in three different office and live in three different houses. Is it possible to find paths for each person from house to office so that no pair of the paths intersect.

3. The Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra states that if $n \geq 1$ and $a_0, \cdots a_{n-1} \in \mathbb{C}$ then there is some $z \in \mathbb{C}$ such that

$$z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0$$

Note that this is not true over the field of real numbers. The FTA is an existence result only - it says nothing about the location of z Many questions about the existence of solutions of equations ar topological.

CHAPTER 1

Set Theory

1. Sets

Topology is typically one of the first courses in which the student is mostly focused on logical and set theoretical concepts, rather than on computational issues. In this chapter we will review some of the basic notions of set theory that will be used in the rest of the course.

1.1. Equality and containment. We will take the notion of a set as a primitive concept. We write $x \in X$ if x is an element of the set X (or if x belongs to the set X).

If X_1 and X_2 are sets, then we write

$$X_1 = X_2$$

if, $x \in X_1 \Leftrightarrow x \in X_2$.

We write $A \subset X$ if every element of A is also an element of X.

1.2. The set builder notation. There are various common ways in which sets are specified or described. One very common way is to describe a set is to give properties that characterise its elements. For example, we might say that X is the set of all integers that are one greater than the square of an integer. Rather than writing out the description as in the previous sentence, it is more common to use mathematical shorthand as follows.

$$X = \{a \in \mathbb{Z} : a = n^2 + 1 \text{ for some } n \in \mathbb{Z}\}.$$

The curly braces "$\{$" and "$\}$" enclose a description of the elements of a set. In this context the colon ":" is shorthand for "such that". We could also specify the same set in the following way.

$$X = \{n^2 + 1 : n \in \mathbb{Z}\}.$$

2. Operations on Sets

The union of sets A and B, denoted $A \cup B$, is defined by

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

The intersection of A and B, denoted $A \cap B$, is defined by

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Sometimes, we would like to discuss the union or intersection of infinitely many sets. Thus, suppose that $\{A_i, i \in I\}$ is a collection of sets. Then $x \in \bigcup_{i \in I} A_i \Leftrightarrow x$ belongs to at least one A_{i_0}. Similarly $x \in \bigcap_{i \in I} A_i \Leftrightarrow x$ belongs to every A_i. In this context, the set I is called an indexing set.

In the special case where the indexing set is a set of consecutive integers, we often use the notations $\bigcup_{i=m}^{i=n} A_i$ or $\bigcap_{i=m}^{i=n} A_i$.

The complement of A in B, denoted $B - A$, is defined by

$$B - A = \{x \in B : x \notin A\}.$$

Given sets A and B, the (cartesian) product of A and B, denoted $A \times B$, is defined by

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

In other words $A \times B$ is the set of all ordered pairs whose first component is an element of A and whose second component is an element of B. Note that in this context (a, b) denotes an ordered pair and not an interval of $\mathbb{R}$.

2.1. Equivalence relations. An equivalence relation on a set X is a subset R of $X \times X$ that satisfies the following three conditions.

(1) For all $x \in X$, $(x, x) \in R$ (i.e. R is reflexive).
(2) If $(x, y) \in R$ then $(y, x) \in R$ (i.e. R is symmetric).
(3) If $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$ (i.e. R is transitive).

If R is an equivalence relation on X, we will write $x \sim_R y$ if $(x, y) \in R$, or sometimes just $x \sim y$ (if no possible confusion can arise from suppressing the R in the notation). If R is an equivalence relation on X, and $x \in X$, then the equivalence class of x, denoted by $[x]_R$ or $[x]$, is defined by

$$[x] = \{y \in X : x \sim y\}.$$

The set of all equivalence classes, denoted by $X/\sim$, is defined by

$$X/\sim = \{[x] : x \in X\}.$$

2.2. Problems.

(1) Give a concise description of the set

$$\bigcup_{i=1}^{\infty} [\frac{1}{n}, 1]$$

where $[a, b]$ denotes $\{x \in \mathbb{R} : a \leq x \leq b\}$.

(2) True or false: If $A \cap B \neq \emptyset$ and $B \cap C \neq \emptyset$ then $A \cap C \neq \emptyset$.
(3) Suppose that $\{A_i : i \in I\}$ is a collection of sets. What is meant by the notation $\prod_{i \in I} A_i$?
(4) Suppose that R_1 and R_2 are both equivalence relations on a set X. Prove that $R_1 \cap R_2$ is also an equivalence relation on X.
(5) Let A be a subset of $X \times X$. Prove that there is a unique equivalence relation R_A with the property that R_A is a subset of any equivalence relation that contains A (note: In this situation, we say that R_A is the equivalence relation generated by A)

3. The Algebra of Sets

THEOREM 3.1. *Suppose that A, B and C are sets. Then*
(1) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(2) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

PROOF. INSERT PROOF HERE □

3.1. De Morgan's Laws. The conclusions of the following theorem are often referred to as De Morgan's Laws.

THEOREM 3.2. *Suppose that X and $A_i, i \in I$ are sets. Then*
(1) $X - \bigcup_{i \in I} A_i = \bigcap_{i \in I}(X - A_i)$
(2) $X - \bigcap_{i \in I} A_i = \bigcup_{i \in I}(X - A_i)$

PROOF. We will prove the first conclusion and leave the proof of the second conclusion to the reader. We have $x \in X - \bigcup_{i \in I} A_i \Leftrightarrow x \in X$ and $x \notin \bigcup_{i \in I} A_i$. But $x \notin \bigcup_{i \in I} A_i \Leftrightarrow x \notin A_i$ for all $i \in I$. Therefore $x \in X - \bigcup_{i \in I} A_i \Leftrightarrow x \in X - A_i$ for all $i \in I$. Therefore $x \in X - \bigcup_{i \in I} A_i \Leftrightarrow x \in \bigcap_{i \in I}(X - A_i)$ as required. □

Informally, De Morgan's Laws say that the complement of the union is the intersection of the complements, and that the complement of the intersection is the union of the complements.

4. Functions

We will also take the notion of function as a primitive notion. We write $f : X \to Y$ if f is a function whose domain is X and whose codomain is Y. Throughout this section we will assume that $f : X \to Y$.

A function f is injective if $f(x) = f(y) \Rightarrow x = y$. A function $f : X \to Y$ is surjective if, for every $y \in Y$, there exists $x \in X$

such that $y = f(x)$. A function is bijective if it is both injective and surjective.

Given that $A \subset X$, the image of A under f, denoted $f(A)$ is defined by

$$f(A) = \{y : y = f(a) \text{ for some } a \in A\}.$$

Given that $B \subset Y$, the preimage of B under f, denoted $f^{-1}(B)$, is defined by

$$f^{-1}(B) = \{x \in X : f(x) \in B\}$$

4.1. Sequences. A sequence in X is a function whose domain is $\mathbb{N}$ and whose codomain is X (i.e. a function $f : \mathbb{N} \to X$). The value of the function at n is called the nth term of the sequence and is usually denoted by f_n (rather than $f(n)$).

We often write things like "$\ldots (x_n)_{n=1}^{\infty}$ is a sequence in ...", which indicates that x_n is the nth term of the given sequence. Sometimes we will just write (x_n) instead of $(x_n)_{n=1}^{\infty}$ when there is no possible confusion.

Suppose that $(x_n)_{n=1}^{\infty}$ is a sequence in X. A subsequence of (x_n) is a sequence of the form $(x_{n_k})_{k=1}^{\infty}$, where (n_k) is a strictly increasing sequence in $\mathbb{N}$. In other words, $n_{i+1} > n_i$ for all $i \in \mathbb{N}$. We can think of a subsequence of (x_n) as the sequence that is left after we delete some of the terms of (x_n) (we can delete infinitely many terms, but we must leave infinitely terms undeleted).

For example, suppose that $x_n = \frac{1}{n}$. So

$$(x_n) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots).$$

Now suppose $n_k = k^2 + 1$. So

$$(n_k) = (2, 5, 10, 17, \ldots).$$

The

$$(x_{n_k}) = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{1}{17}, \ldots).$$

4.2. Problems.

(1) Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(a) = \sqrt{a}$. What is $f^{-1}(\mathbb{Q})$?

(2) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is quadratic polynomial function.
   (a) Prove that f is not surjective.
   (b) Prove that f is not injective.

(3) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^3 + ax^2 + bx + c$ where a, b and c are real constants. Prove that g is bijective if and only if $a^2 < 3b$.

(4) Let X be a set and let $X^{\mathbb{N}}$ denote the set of all sequences in X. Show that there is a natural bijective function

$$X^{\mathbb{N}} \to X \times X \times X \times \dots$$

CHAPTER 2

Topological Spaces

1. Motivation

Suppose that (X, d) is a metric space (i.e. d is a metric on the set X). Recall that a subset $A \subset X$ is said to be d-open in X if, given any $x \in A$, there exists $\epsilon > 0$ such that $d(x, z) < \epsilon \Rightarrow z \in A$. This notion of "openness" of a subset plays a crucial role in metric space theory. So it is natural to ask if it is possible to axiomatise this notion of "openness". In other words, is it possible to write down a list of properties which somehow capture all the 'essential' features of d-open subsets of X?

2. The Definition and Some First Examples

Definition 2.1. Let X be a set. We say that $\mathcal{T}$ is a topology on X if $\mathcal{T}$ is a set of subsets of X that satisfies the following properties.

(1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
(2) If $G_i \in \mathcal{T}$ for all $i \in I$ then $\bigcup_{i \in I} G_i \in \mathcal{T}$.
(3) If $G_1 \in \mathcal{T}$ and $G_2 \in \mathcal{T}$, then $G_1 \cap G_2 \in \mathcal{T}$.

The elements of $\mathcal{T}$ are called open subsets of X (with respect to $\mathcal{T}$). A topological space is a pair $(X, \mathcal{T})$ where $\mathcal{T}$ is a topology on the set X.

Informally, we can summarise the properties of a topology by saying that any union of open sets is open and that the intersection of two open sets is open.

Example 2.2. Let $X = \{a, b, c\}$ and let $\mathcal{T} = \{\emptyset, \{a\}\{a, b\}, \{a, c\}, X\}$. Then $\mathcal{T}$ is a topology on X.

Example 2.3. Let $P(X)$ denote the set of all subsets of a set X. Then $P(X)$ is a topology on X. This topology is called the discrete topology.

Example 2.4. Let X be a set and let

$$X_{fc} = \{G \subset X : X - G \text{ is finte}\} \cup \{\emptyset\}$$

We will show that X_{fc} is a topology on X. Clearly $\emptyset \in X_{fc}$. Also $X - X = \emptyset$ which is finite, so $X \in X_{fc}$.

Now suppose that $G_i \in X_{fc}$ for $i \in I$. If $\bigcup_{i\in I} G_i \neq \emptyset$ then $G_{i_0} \neq \emptyset$ for some $i_0 \in I$. So $X - G_{i_0}$ is finite. We have

$$\begin{aligned} X - \textstyle\bigcup_{i\in I} G_i &= \textstyle\bigcap_{i\in I}(X - G_i) \quad \text{by de Morgan's Laws} \\ &\subset X - G_{i_0} \end{aligned}$$

Thus $X - \bigcup_{i\in I} G_i$ is finite (since it is a subset of a finite set), and $\bigcup_{i\in I} G_i \in X_{fc}$.

Suppose that $G_1 \in X_{fc}$ and $G_2 \in X_{fc}$. If either of G_1 or G_2 is the empty set, then so is $G_1 \cap G_2$. So suppose that $G_1 \neq \emptyset$ and $G_2 \neq \emptyset$. Then

$$X - (G_1 \cap G_2) = (X - G_1) \cup (X - G_2)$$

by de Morgan's Laws. Therefore $X - (G_1 \cap G_2)$ is finite (since it is the union of two finite sets), and $G_1 \cap G_2 \in X_{fc}$ as required.

The topology X_{fc} is called the finite complement topology on X.

2.1. The topology induced by a metric. Let (X, d) be a metric space. Recall that a subset G of X is said to be d-open if, for all $x \in G$ there exists $\epsilon > 0$ such that $d(x, y) < \epsilon \Rightarrow y \in G$. Let

$$\mathcal{T}_d = \{G \subset X : G \text{ is } d\text{-open}\}$$

LEMMA 2.5. *$\mathcal{T}_d$ is a topology on X.*

PROOF. It is clear that $\emptyset$ and X are both in $\mathcal{T}_d$.

Now suppose that G_i is d-open for $i \in I$. Let $x \in \bigcup_{i\in I} G_i$. So $x \in G_{i_0}$ for some $i_0 \in I$. Therefore, there is some $\epsilon > 0$ such that $d(x, y) < \epsilon \Rightarrow y \in G_i$. Clearly $d(x, y) < \epsilon \Rightarrow y \in \bigcup_{i\in I} G_i$, so $\bigcup_{i\in I} G_i$ is d-open.

Suppose that $G_1 \in \mathcal{T}_d$ and $G_1 \in \mathcal{T}_d$. Let $x \in G_1 \cap G_2$. There is some $\epsilon_1 > 0$ such that $d(x, y) < \epsilon_1 \Rightarrow y \in G_1$. Similarly there is some $\epsilon_2 > 0$ such that $d(x, y) < \epsilon_2 \Rightarrow y \in G_2$. Now, let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Clearly, $\epsilon > 0$ and $d(x, y) < \epsilon \Rightarrow y \in G_1 \cap G_2$. So $G_1 \cap G_2$ is d-open.

We have shown that unions of d-open subsets are d-open and that the intersection of any two d-open subsets is d-open. Therefore $\mathcal{T}_d$ is a topology on X. $\square$

The topology $\mathcal{T}_d$ is called the topology induced by the metric d, or sometimes just the metric topology.

EXAMPLE 2.6. The standard metric on $\mathbb{R}^n$ is defined by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are points in $\mathbb{R}^n$. The topology induced by this metric is called the standard topology on $\mathbb{R}^n$, or the usual topology on $\mathbb{R}^n$.

3. Bases

DEFINITION 3.1. Let $\mathcal{B}$ be a collection of subsets of X. We say that $\mathcal{B}$ is basis (for a topology on X) if the following two statements are true.

(1) $\bigcup_{B\in\mathcal{B}} B = X$.
(2) Given $B_1 \in \mathcal{B}$ and $B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there is some $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$,

Elements of $\mathcal{B}$ are called basic (open) subsets of X.

Note that the second property in Definition 3.1 is a weaker condition than being closed under finite intersection. In other words, any collection that closed under finite intersections also has this property.

EXAMPLE 3.2. Let

$$\mathcal{B} = \{(a,b) \text{ where } a \text{ and } b \text{ are real numbers with } a < b\}.$$

Then $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$.

PROPOSITION 3.3. *Let $\mathcal{B}$ be a basis as defined above. Then the collection*

$$\mathcal{T} = \{G \subset X : G \textit{ is a union of basic subsets}\} \cup \{\emptyset\}$$

is a topology on X.

PROOF. By construction $\emptyset \in \mathcal{T}$. Also, by the definition of a basis X is a union of basic subsets. Therefore $X \in \mathcal{T}$.

Now suppose that, for each $i \in I$, $G_i \in \mathcal{T}$. So each G_i is a union of basic subsets of X. Then clearly $\bigcup_{i\in I} G_i$ is also a union of basic subsets of X. So $\bigcup_{i\in I} G_i \in \mathcal{T}$.

Finally, suppose that $G_1 \in \mathcal{T}$ and $G_2 \in \mathcal{T}$. Let $x \in G_1 \cap G_2$. Since $x \in G_1$, there is some basic subset B_1 such that $x \in B_1$ and $B_1 \subset G_1$. Similarly, there is some basic subset B_2 such that $x \in B_2$ and $B_2 \subset G_2$. Now $x \in B_1 \cap B_2$, so by the definition of a basis there is some basic set B_x such that $x \in B_x$ and $B_x \subset B_1 \cap B_2$. But $B_1 \cap B_2 \subset G_1 \cap G_2$. So for each $x \in G_1 \cap G_2$ there is a basic set B_x such that $x \in B_x$ and $B_x \subset G_1 \cap G_2$. Clearly,

$$G_1 \cap G_2 = \bigcup_{x\in G_1\cap G_2} B_x,$$

so $G_1 \cap G_2 \in \mathcal{T}$ as required. □

We say that the topology described in Proposition 3.3 is the topology generated by the basis $\mathcal{B}$.

EXAMPLE 3.4. (The Sorgenfrey topology) Consider the following collection of subsets of $\mathbb{R}$.

$$\mathcal{B} = \{[a, b) : a < b\}.$$

Recall that $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$. We will show that $\mathcal{B}$ is a basis. Clearly, $\mathbb{R} = \bigcup_{n=1}^{\infty}[-n, n)$. Now, suppose that $[a_1, b_1) \in \mathcal{B}$ and $[a_2, b_2) \in \mathcal{B}$. It is easy to check that $[\max a_1, a_2, \min\{b_1, b_2\}) = [a_1, b_1) \cap [a_2, b_2)$ (the reader should verify this). So in fact $\mathcal{B}$ is closed under finite intersections.

The topology generated by this basis is called the Sorgenfrey topology on $\mathbb{R}$.

3.1. Problems.

(1) Prove that $(0, 1)$ is an open subset of $\mathbb{R}$ with respect to the Sorgenfrey topology.
(2) Prove that every subset of $\mathbb{R}$ that is open with respect to the usual topology is also open with respect to the Sorgenfrey topology.
(3) Let (X, d) be a metric space. Recall that if $x \in X$ and $r \in \mathbb{R}$, then $B(x, r) = \{y \in X : d(x, y) < r\}$. Let

$$\mathcal{B} = \{B(x, r) : x \in X, r \in \mathbb{R}\}.$$

Show that $\mathcal{B}$ is a basis. Show that the topology generated by this basis is the same as the topology described in Section 2.1.
(4) Let X be a set and let $\mathcal{B} = \{\{x\} : x \in X\}$. Show that $\mathcal{B}$ is a basis. What is the topology generated by $\mathcal{B}$?

4. Closed Sets

Let $(X, \mathcal{T})$ be a topological space. A closed subset of X is a subset of X whose complement in X is open, In other words A is a closed subset of X if and only if $X - A \in \mathcal{T}$.

For example, the interval $[a, b]$ is a closed subset of $\mathbb{R}$ with respect to the usual topology, since $[a, b] = \mathbb{R} - ((-\infty, a) \cup (b, \infty))$.

Using de Morgan's Laws, we can translate many of the properties of open sets into corresponding properties of open sets. For example

PROPOSITION 4.1. *Let $(X, \mathcal{T})$ be a topological space. Then*

(1) *$\emptyset$ and X are closed subsets of X.*
(2) *If, for each $i \in I$, is a closed subset of X, then $\bigcap_{i \in I} A_i$ is a closed subset of X.*
(3) *If A_1 and A_2 are closed subsets of X, then $A_1 \cup A_2$ is a closed subset of X.*

PROOF. □

EXAMPLE 4.2. (The Cantor Set) For each positive integer n, let

$$A_n = \bigcup_{j=0}^{\frac{1}{2}(3^n-1)} \left[\frac{2j}{3^n}, \frac{2j+1}{3^n}\right].$$

Now, for each $n \in \mathbb{N}$, A_n is a closed subset of $\mathbb{R}$ with respect to the usual topology, since it is a finite union of closed intervals. We define the Cantor ternary set by letting

$$C = \bigcap_{n=1}^{\infty} A_n$$

Since each A_n is closed, we see that C is also a closed subset of $\mathbb{R}$ with respect to the usual topology. The reader should try to prove that C consists of all real numbers in $[0, 1]$ that cannot have the digit 2 occurring in any base 3 expansion of that number (note that some numbers may have non unique base 3 expansions).

A topological space is said to be T_1 if every singleton subset is a closed subset. Note that we have already seen some examples of spaces that are not T_1. However, $\mathbb{R}^n$ with its usual topology is certainly T_1. Indeed, any metric space is T_1.

5. Interior and Closure

Throughout this section $(X, \mathcal{T})$ is a topological space.

DEFINITION 5.1. Let A be a subset of X. The interior of A, denoted by $\mathring{A}$ or $\operatorname{int}(A)$ is the union of all of the open sets, contained in A. The closure of A, denoted $\overline{A}$ or $\operatorname{cl}(A)$ is the intersection of all of the closed sets that contain A.

EXAMPLE 5.2. Consider $\mathbb{R}$ with the usual topology. Then $\mathring{\mathbb{Q}} = \emptyset$ and $\overline{\mathbb{Q}} = \mathbb{R}$

EXAMPLE 5.3. Consider $\mathbb{R}$ with the Sorgenfrey topology. Then $\operatorname{int}([0, 1)) = [0, 1)$ and $\operatorname{cl}([0, 1)) = [0, 1)$.

We say that a subset A of X is dense in X if $\overline{A} = X$.

Now, we would like to characterise the points that lie in the interior or closure of a set A.

PROPOSITION 5.4. *Let A be a subset of X. Then*

(1) *$x \in \mathring{A}$ if and only if there is some open subset B such that $x \in B$ and $B \subset A$.*

(2) *$x \in \overline{A}$ if and only if every open subset that contains x intersects A nontrivially. (i.e. B open and $x \in B$, $\Rightarrow B \cap A \neq \emptyset$).*

Proof. By definition $\mathring{A} = \bigcup_{G \subset A, G \text{open}} G$, so (1) follows immediately from the definition. For (2) observe first that, given subsets A and C of X, we have $A \subset C \Leftrightarrow X - C \subset X - A$. Therefore we can argue that $x \in \overline{A} \Leftrightarrow x$ belongs to every closed set that contains $A \Leftrightarrow x$ does not belong to any open set contained in $X - A \Leftrightarrow x$ does not belong to any open subset of X that is disjoint from $A \Leftrightarrow$ any open set that contains x is not disjoint from A. (Recall that sets are disjoint if and only if they have empty intersection.) □

Now, we consider the case where the topology is induced by a metric

Theorem 5.5. *Let (X, d) be a metric space and suppose that $A \subset X$. Then $x \in \overline{A}$ if and only if there is a sequence in A that converges to x.*

Proof. Suppose first that $x \in \overline{A}$. Let $n \in \mathbb{N}$ and recall that $B(x, \frac{1}{n}) = \{y \in X : d(x, y) < \frac{1}{n}\}$ is an open subset of X that contains x. By Proposition 5.4 $A \cap B(x, \frac{1}{n}) \neq \emptyset$, so we can choose $a_n \in A \cap B(x, \frac{1}{n})$. It is clear that (a_n) is a sequence in A that converges to x.

On the other hand, suppose that (a_n) is a sequence in A that converges to x and let G be any open set containing x. There is some $\epsilon > 0$ such that $d(x, y) < \epsilon \Rightarrow y \in G$. However, there is some N such that $n > N \Rightarrow d(a_n, x) < \epsilon$. Therefore $a_n \in G \cap A$ for all $n > N$. In particular $G \cap A \neq \emptyset$. So, by Proposition 5.4, $x \in \overline{A}$. □

5.1. Problems.

(1) Prove that $X - \mathrm{int}(A) = \mathrm{cl}(X - A)$.
(2) Prove that $X - \mathrm{cl}(A) = \mathrm{int}(X - A)$.
(3) Prove that $\mathring{A} = A \Leftrightarrow A$ is an open subset of X.
(4) True or false: $\overline{A \cap B} = \overline{A} \cap \overline{B}$.
(5) True or false: $\mathrm{int}(A \cap B) = \mathring{A} \cap \mathring{B}$.
(6) True or false: $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(7) True or false: $\mathrm{int}(A \cup B) = \mathring{A} \cup \mathring{B}$.

CHAPTER 3

Continuity and Homeomorphisms

1. Continuous Functions

So far we have been concerned with topological spaces 'in isolation'. Now we want to consider functions between such spaces. The topology allows us to talk about continuity of such functions. Recall that if $f : X \to Y$ is a function and $B \subset Y$, then $f^{-1}(B) = \{x \in X : f(x) \in B\}$ ($f^{-1}(B)$ is called the preimage of B under f.)

Definition 1.1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $f : X \to Y$ be a function. We say that f is a continuous function if, for every open subset B of Y, its preimage $f^{-1}(B)$ is an open subset of X.

Notice, the relative simplicity of Definition 1.1 as compared to the typical ϵ-δ definition of continuity that one sees in a first course in analysis or in metric spaces. It can be distilled down to the simple mnemonic "continuous means preimage of open is open". By contrast, the ϵ-δ definition of continuity in metric spaces is awkward to state and to use.

Of course, we must do some work to prove that this definition of continuity is really an extension of the notion of continuity of functions of metric spaces. That is the content of Theorem 1.4 below. Before proving that, we give some examples to illustrate Definition 1.1.

Example 1.2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x$. In this case, if $B \subset \mathbb{R}$ then $f^{-1}(B) = B$. So f is continuous if and only if every set that is open with respect to the topology of the codomain is also open with respect to the topology of the domain. For example, if the domain has the usual topology and the codomain has the Sorgenfrey topology, then f is not continuous. However, if the domain has the Sorgenfrey topology and the codomain has the usual topology then f is continuous (why?)

Example 1.3. Let $\mathbb{F}$ be a field and let $p : \mathbb{F} \to \mathbb{F}$ be a polynomial function. Suppose that the domain and codomain are equipped with the finite complement topology, then p is continuous. Note that continuity of polynomial functions in this context is a consequence of the purely algebraic fact that nontrivial polynomials have finitely many roots. Indeed this example has a far reaching generalisation in the subject of algebraic geometry, where one considers the so called Zariski topology and continuous functions with respect to that topology.

Now, we return to the comparison of continuity in the metric and topological contexts.

THEOREM 1.4. *Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$. Then f is continuous is the sense of metric space theory if and only if f is continuous in the sense of Definition 1.1 with respect to topologies induced by the metrics.*

PROOF. First suppose that f is continuous in the metric sense. Let B be an open subset of Y and let $x \in f^{-1}(B)$. Since B is open there is some $\epsilon > 0$ such that $d_Y(f(x), y) < \epsilon \Rightarrow y \in B$. Now, since f is continuous in the metric sense, there is some $\delta > 0$ such that $d_X(x, u) < \delta \Rightarrow d_Y(f(x), f(u)) < \epsilon$. Therefore, if $d_X(x, u) < \delta$ then $u \in f^{-1}(B)$. Thus $f^{-1}(B)$ is an open subset of X.

Conversely, suppose that f is continuous in the sense of Definition 1.1. Let $x \in X$ and let $\epsilon > 0$. The set $B = \{y \in Y : d_Y(f(x), y) < \epsilon\}$ is an open subset of Y, so, by assumption, $f^{-1}(B)$ is an open subset of X. Now, clearly $x \in f^{-1}(B)$, so there is some $\delta > 0$ such that $d_X(x, u) < \delta \Rightarrow u \in f^{-1}(B)$. Therefore, $d_X(x, u) < \epsilon \Rightarrow d_Y(f(x), f(u)) < \epsilon$ and f is continuous in the sense of metric spaces, as required. $\square$

We observe that we can characterise also continuity using closed sets

PROPOSITION 1.5. *A function $f : X \to Y$ is continuous if and only if the preimage of every closed subset of Y is a closed subset of X.*

1.1. Problems.

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. In each of the following cases, decide whether or not f is continuous.
  (a) The domain has the usual topology and the codomain has the discrete topology.
  (b) The domain has the discrete topology and the codomain has the Sorgenfrey topology.
  (c) The domain has the Sogenfrey topology and the codomain has the usual topology.
  (d) The domain has the usual topology and the codomain has the finite complement topology.
  (e) The domain has the finite complement topology and the codomain has the usual topology.

(2) Prove that a function $f : X \to Y$ is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for all subsets A of X.

2. Homeomorphisms

In popular literature, topology is sometimes referred to as "rubber sheet geometry". The following definition is the rigourous idea that underlies that intuitive notion.

Definition 2.1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A function $f : X \to Y$ is a homeomorphism if f is continuous, bijective and has a continuous inverse. If a homeomorphism $f : X \to Y$ exists then we say that $(X, \mathcal{T}_X)$ is homeomorphic to $(Y, \mathcal{T}_Y)$ and we write $(X, \mathcal{T}_X) \equiv (Y, \mathcal{T}_Y)$ (or sometimes just $X \equiv Y$).

If $X \equiv Y$ then X and Y are indistinguishable as topological spaces. One of the fundamental goals of topology is to develop methods for deciding when two spaces are related by a homeomorphism. In general, it can be very difficult to decide.

Example 2.2. Consider the metric spaces $(0, 1)$ and $(0, \infty)$ where the metric in each case is the standard metric (i.e. $d(x, y) = |x - y|$ The function $f : (0, 1) \to (0, \infty)$ defined by $f(x) = \frac{1}{x} - 1$ is a homeomorphism. Thus $(0, 1) \equiv (0, \infty)$. Observe that as metric spaces, these spaces are fundamentally different. Indeed the standard metric on $(0, 1)$ is bounded whereas the standard metric on $(0, \infty)$ is unbounded. However as topological spaces, these two spaces are indistinguishable.

Proposition 2.3. *The relation of homeomorphism is an equivalence relation.*

Proof. We must show that "$\equiv$" is reflexive, symmetric and transitive.

Let $\mathbb{I}_X : X \to X$ denote the identity function (i.e. $\mathbb{I}_X(x) = x$ for all $x \in X$). Clearly, $\mathbb{I}_X$ is a homeomorphism. So $X \equiv X$.

Suppose that $X \equiv Y$. So there is a homeomorphism $f : X \to Y$. It is easy to check that $f^{-1} : Y \to X$ is also a homeomorphism. So $Y \equiv X$.

Finally, suppose that $X \equiv Y$ and $Y \equiv Z$. So there are homeomorphisms $f : X \to Y$ and $g : Y \to Z$. Now, it is clear that $g \circ f : X \to Z$ is a homeomorphism, so $X \equiv Z$. $\square$

Example 2.4. Consider $(0, \infty)$ with its usual topology and $(0, 1)$ with its usual topology. The function $f : (0, 1) \to (0, \infty)$ given by $f(t) = \frac{1}{t} - 1$ is a homeomorphism.

EXAMPLE 2.5. On the other hand $(\mathbb{R}, \mathcal{U})$ is not homeomorphic to $(\mathbb{R}, \mathcal{P})$ where $\mathcal{U}$ is the usual topology and $\mathcal{P}$ is the discrete topology. This follows from the fact that if $f : \mathbb{R} \to \mathbb{R}$ is continuous with respect to the usual topology on the domain and the discrete topology on the codomain, then f is a constant function and therefore is not a bijection.

2.1. Problems.

(1) Let $\mathcal{U}$ be the usual topology on $[0, 2]$ and define

$$\mathcal{B} = \mathcal{U} \cup \{[1, a) : a > 1\}.$$

 (a) Show that $\mathcal{B}$ is a basis for a topology on $[0, 2]$. Denote this topology by $\mathcal{T}$.

 (b) Prove that $([0, 2], \mathcal{T}) \equiv [0, 1) \cup [2, 3]$ where $[0, 1) \cup [2, 3]$ has the usual topology.

CHAPTER 4

New Spaces From Old

1. Subspaces

Let $(X, \mathcal{T})$ be a topological space and let A be a subset of X. We define

$$\mathcal{T}_A = \{G \cap A : G \in \mathcal{T}\}.$$

PROPOSITION 1.1. *The collection $\mathcal{T}_A$ is a topology on A.*

PROOF. We verify that $\mathcal{T}_A$ has all the properties required of a topology as follows. First $\emptyset = \emptyset \cap A$ and $A = X \cap A$. Therefore $\emptyset \in \mathcal{T}_A$ and $A \in \mathcal{T}_A$.

Now suppose that $U_i \in \mathcal{T}_A$ for $i \in I$. So for each $i \in I$ there is some open subset G_i of X such that $U_i = G_i \cap A$. Now

$$\bigcup_{i \in I} U_i = \bigcup_{i \in I} (G_i \cap A) = (\bigcup_{i \in I} G_i) \cap A \in \mathcal{T}_A.$$

Finally, suppose that U_1 and U_2 are in $\mathcal{T}_A$. So $U_1 = G_1 \cap A$ and $U_2 = G_2 \cap A$ for some open subsets G_1 and G_2 of X. Then

$$U_1 \cap U_2 = (G_1 \cap A) \cap (G_2 \cap A) = (G_1 \cap G_2) \cap A \in \mathcal{T}_A.$$

$\square$

The topology $\mathcal{T}_A$ is called the induced topology, or the subspace topology, on A.

EXAMPLE 1.2. Consider $\mathbb{R}$ with its usual topology. The induced topology on the subset $\mathbb{Z}$ is the discrete topology. We can prove this by observing that for any $n \in \mathbb{Z}$, we have

$$\{n\} = (n - \frac{1}{2}, n + \frac{1}{2}) \cap \mathbb{Z}.$$

Therefore $\{n\}$ is an open subset of $\mathbb{Z}$ with respect to the subspace topology. However, if every singleton subset is open, then the topology must be the discrete topology.

EXAMPLE 1.3. Again consider $\mathbb{R}$ with the usual topology and consider the subset $X = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ and $Y = X \cup \{0\}$. the induced topology on X is the discrete topology. The induced topology on Y

is not the discrete topology, since any open subset of Y containing 0, must have finite complement in Y.

2. Quotient Spaces

It is recommended that the reader review the section on equivalence relations (Section 2.1) before proceeding.

We begin by observing that given a set X and a subset $A \subset X$, there is an injective function $i : A \to X$ defined by $i(x) = x$ called the inclusion function. In a sense, every injective function can be thought of as an inclusion function.

In a similar (but dual) way, surjective functions correspond to equivalence relations. Given a function $p : Y \to B$, we can define an equivalence relation on Y by $x \sim_p y \Leftrightarrow p(x) = p(y)$. We leave it as an easy exercise to check that $\sim_p$ is indeed an equivalence relation. On the other hand, given an equivalence relation on a set Y, we have the canonical function

$$p : Y \to Y/\sim$$

defined by $p(y) = [y]$. This function is sometimes called the quotient map.

2.1. The quotient topology. Throughout this section, let $(X, \mathcal{T})$ be a topological space. Suppose that $p : X \to A$ is a function (of sets). Define

$$\mathcal{T}_p = \{G \subset A : p^{-1}(G) \in \mathcal{T}\}.$$

LEMMA 2.1. *The collection $\mathcal{T}_p$ forms a topology on A.*

PROOF. First, observe that $\emptyset = p^{-1}(\emptyset)$ and $X = p^{-1}(A)$, so $\emptyset \in \mathcal{T}_p$ and $A \in \mathcal{T}_p$.

Now suppose that $G_i \in \mathcal{T}_p$ for $i \in I$. So $p^{-1}(G_i) \in \mathcal{T}$ for all $i \in I$. Then $p^{-1}(\bigcup_{i\in I} G_i) = \bigcup_{i\in I} p^{-1}(G_i) \in \mathcal{T}$, since $\mathcal{T}$ is closed under unions. Therefore $\bigcup_{i\in I} G_i \in \mathcal{T}_p$.

Similarly, if $G_1 \in \mathcal{T}_p$ and $G_2 \in \mathcal{T}_p$. Then $p^{-1}(G_1 \cap G_2) = p^{-1}(G_1) \cap p^{-1}(G_2) \in \mathcal{T}$, since $p^{-1}(G_1) \in \mathcal{T}$ and $p^{-1}(G_2) \in \mathcal{T}$.. Therefore $G_1 \cap G_2 \in \mathcal{T}_p$. $\square$

The topology $\mathcal{T}_p$ is called the quotient topology on X.

LEMMA 2.2. *With notation as above, the function $p : X \to A$ is a continuous function, with respect to the topologies $\mathcal{T}$ and $\mathcal{T}_p$ on the domain and codomain respectively.*

PROOF. $\square$

It is useful to be able to think about the quotient topology in the context of equivalence relations as well. Suppose that $\sim$ is an equivalence relation on X and let $p : X \to X/\sim$ be the quotient map (as described above). In this case, the inverse image under p of a subset G of $X/\sim$ is just the union of all the equivalence classes of $\sim$ that are members of G. Thus, we see that the quotient topology can be described as follows.

$$\mathcal{T}_p = \{G \subset X/\sim : \bigcup_{[x]\in G} [x] \text{ is an open subset of } X\}.$$

In this context, we often refer to the topological space $(X/\sim, \mathcal{T}_p)$ as the quotient space. The quotient space construction is a very important way of constructing examples of topological spaces with interesting properties.

Example 2.3. Define an equivalence relation on $\mathbb{R}$ by $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$. The quotient space $\mathbb{R}/\sim$ is homeomorphic to $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. The proof of this fact will be given below.

Example 2.4. Consider $\mathbb{R}^{n+1} - \{0\}$. We can define an equivalence relation on this space by $(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n)$ if and only if there is some $\lambda \neq 0$ such that $(y_0, \ldots, y_n) = \lambda(x_0, \ldots, x_n)$. We leave it to the reader to check that this is indeed an equivalence relation. We define

$$\mathbb{RP}^n = \frac{\mathbb{R}^{n+1} - \{0\}}{\sim}.$$

If $\mathbb{R}^{n+1} - \{0\}$ is equipped with its usual topology, then $\mathbb{RP}^n$ can be topologised as a quotient space. This space is called the n-dimensional (real) projective space.

Now suppose that $\sim$ is an equivalence relation on X. Let $(B, \mathcal{S})$ be a topological space and suppose that $f : X \to B$ has the property that $f(x) = f(y)$ whenever $x \sim y$. In this situation, we can define a function $\overline{f} : X/\sim \to B$, by $\overline{f}([x]) = f(x)$. The function $\overline{f}$ is said to be induced by f.

Theorem 2.5. *With the notation as above, if f is continuous, then $\overline{f}$ is continuous with respect to the quotient topology on $X/\sim$.*

Proof. Suppose that U is an open subset of B and consider its preimage $\overline{f}^{-1}(U)$. We must show that this an open subset of $X/\sim$. Now, by construction $f = \overline{f} \circ p$. Therefore $f^{-1}(U) = p^{-1}(\overline{f}^{-1}(U))$. But f is continuous. Therefore $f^{-1}(U)$ is an open subset of X. Therefore $p^{-1}(\overline{f}^{-1}(U))$ is an open subset of X. So by the definition of the quotient topology $\overline{f}^{-1}(U)$ is an open subset of $X/\sim$. Therefore $\overline{f}$ is a continuous function. $\square$

Now, we can prove the assertion of Example 2.3. Define $f : \mathbb{R} \to S^1$ by $f(x) = (\cos(2\pi x), \sin(2\pi x))$. It is clear that $f(x) = f(y) \Leftrightarrow x - y \in \mathbb{Z} \Leftrightarrow x \sim y$ where $\sim$ is the equivalence relation defined in Example 2.3. Thus f induces a function $\overline{f} : \mathbb{R}/\sim \to S^1$. By Theorem 2.5 $\overline{f}$ is continuous. Moreover $\overline{f}$ is surjective since f is surjective and $f = \overline{f} \circ p$. Also $\overline{f}$ is injective since $\overline{f}([x]) = \overline{f}([y]) \Rightarrow (\cos(2\pi x), \sin(2\pi x)) = (\cos(2\pi y), \sin(2\pi y)) \Rightarrow x - y \in \mathbb{Z} \Rightarrow [x] = [y]$. So $\overline{f}$ is a continuous bijection. It remains to show that $\overline{f}^{-1}$ is continuous. That is, we must show that if G is an open subset of $\mathbb{R}/\sim$ then $\overline{f}(G)$ is an open subset of S^1. We can do this as follows. Since G is open in $\mathbb{R}/\sim$, it must be that $U = p^{-1}(G)$ is an open subset of $\mathbb{R}$. But $f(U) = \overline{f}(G)$. Now, we claim that $f(U)$ is an open subset of S^1. This follows from the fact that the image under f of any open interval of $\mathbb{R}$ is an open subset of S^1 - since f maps basic open subsets to open subsets - it maps any subset to an open subset (see problems below).

2.2. Problems.

(1) Let X and Y be topological spaces and let $\mathcal{B}_X$ be a basis for the topology on X and let $\mathcal{B}_Y$ be a basis for the topology on Y. Prove the following assertions
   (a) A function $f : X \to Y$ is continuous if and only if, for every basic open subset G of Y, $f^{-1}(G)$ is an open subset of X.
   (b) Suppose that for every basic open subset U of X, $f(U)$ is an open subset of Y. Then, for any open subset V of X, $f(V)$ is an open subset of Y.

3. Product Spaces

Throughout this section, let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Recall that the cartesian product of the sets X and Y is the set

$$X \times Y = \{(x, y) : x \in X, y \in Y\}.$$

In other words, $X \times Y$ is the set of all possible ordered pairs where the first element of the pair comes from X and the second element of the pair comes from Y.

Given that X and Y are equipped with topologies, it seems natural to wonder if there is an obvious way to topologise $X \times Y$. Let $\mathcal{B} = \{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$ (in other words $\mathcal{B}$ consist of all subsets

$X \times Y$ that are products of an open subset of X and an open subset of Y. One might hope that this forms a topology on $X \times Y$. However, we leave it to the reader to check that $\mathcal{B}$ is not generally closed under unions and is therefore not a topology. However,

PROPOSITION 3.1. *The collection $\mathcal{B}$ forms a basis for a topology on $X \times Y$.*

PROOF. We show that $\mathcal{B}$ satisfies all the conditions required by Definition 3.1.

First $X \in \mathcal{T}_X$ and $Y \in \mathcal{T}_Y$. Therefore $X \times Y \in \mathcal{B}$. Also $\emptyset = \emptyset \times \emptyset \in \mathcal{B}$.

Now suppose that $U_1 \times V_1 \in \mathcal{B}$ and $U_2 \times V_2 \in \mathcal{B}$. Then $U_1 \times V_1 \cap U_2 \times V_2 = (U_1 \cap U_2) \times (V_1 \cap V_2)$. Therefore $U_1 \times V_1 \cap U_2 \times V_2 \in \mathcal{B}$. In particular, condition (2) of Definition 3.1 is satisfied. $\square$

The topology generated by the basis $\mathcal{B}$ is called the product topology on $X \times Y$.

EXAMPLE 3.2. Let $\mathcal{U}$ be the usual topology on $\mathbb{R}$. Then the product topology on $\mathbb{R} \times \mathbb{R}$ is the same as the usual topology on $\mathbb{R} \times \mathbb{R}$.

3.1. Problems.

(1) Let $\mathcal{P}_X$ and $\mathcal{P}_Y$ be the discrete topologies on X and Y respectively. Show that the product topology on $X \times Y$ is also the discrete topology on $X \times Y$.

4. Interesting Examples of Spaces

In this section, we consider various examples of topological spaces that arise as special cases of the constructions we have described above (subspaces, quotient spaces and product spaces) and we see that the same topological space can often arise naturally in more than one way. We will not prove the existence of many of the homeomorphisms whose existence are asserted in this section. It is left to the reader to verify their existence.

4.1. Surfaces. Recall that $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. We assume that S^1 is equipped with the topology it inherits as a subspace of $\mathbb{R}^2$. We can define the torus, Σ, to be the product space $S^1 \times S^1$. The normal way to visualise the torus is as a subspace of $\mathbb{R}^3$. Thus, let

$$X = \{((3 + \cos t) \cos s, (3 + \cos t) \sin s, \sin t) \in \mathbb{R}^3 : s, t \in \mathbb{R}\}$$

and suppose that X is topologised as a subspace of $\mathbb{R}^3$. Then $\Sigma \equiv X$.

Another common way to construct the torus is to start with the unit square

$$I^2 = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

and define an equivalence relation on I^2 as follows: Let $\sim$ be the equivalence relation generated by the following relations. $(0, y) \sim (1, y)$ and $(x, 0) \sim (x, 1)$. Now $\Sigma \equiv I^2/\sim$.

Motivated by the last construction of Σ, we can construct other examples of surfaces by gluing the edges of polygons as indicated in the diagram below.

4.2. Topological Groups. Recall that a group is a set G together with a group multiplication that satisfies certain properties. The group multiplication gives a function $\mu : G \times G \to G$ defined by $\mu(g, h) = gh$ and the existence of unique inverses gives a function $\iota : G \to G$ defined by $\iota(g) = g^{-1}$. If G comes equipped with a topology, then $G \times G$ has the product topology. A topological group is a group for which the functions μ and ι are continuous. For example, consider additive group of real numbers. This is a topological group. Another classical example is the group $GL(\mathbb{R}, n)$ consisting of $n \times n$ invertible matrices with real entries. This set is a subspace of $\mathbb{R}^{n^2}$ and so inherits the subspace topology. The group operation is matrix multiplication and we leave it to the reader to prove that multiplication and inversion in this case are both continuous functions.

There is a vast and important theory of topological groups. These objects arise naturally is many areas of applied and pure mathematics.

CHAPTER 5

Connectedness and Compactness

Throughout this chapter $(X, \mathcal{T})$ is a topological space.

1. Connectedness

Definition 1.1. We say that X is connected if no proper subset of X is both closed and open. We say that X is disconnected if there is a proper subset of X that is both closed and open.

Equivalently, X is disconnected if there are disjoint open nonempty subsets A and B such that $X = A \cup B$. We call such a pair $\{A, B\}$ a disconnection of X. So X is connected if there is no such disconnection.

Example 1.2. Consider the subspace $[0, 1] \cup [2, 3]$ of $\mathbb{R}$ (with its usual topology). Then $\{[0, 1], [2, 3]\}$ is a disconnection of this space.

Example 1.3. The space of rational numbers $\mathbb{Q}$ (a subspace of $\mathbb{R}$) is disconnected, since $(-\infty, \sqrt{2}) \cap \mathbb{Q}$ is a subset of $\mathbb{Q}$ that is both closed and open (why is it closed?).

Example 1.4. The space $\mathbb{R}$ (with its usual topology is connected. To see this, suppose that $\{A, B\}$ is a disconnection of $\mathbb{R}$. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} -1 & \text{if } x \in A \\ 1 & \text{if } x \in B \end{cases}$$

For $C \subset \mathbb{R}$, $f^{-1}(C)$ is either $\emptyset$, A, B or $\mathbb{R}$, all of which are open subsets of $\mathbb{R}$. Therefore f is continuous. But clearly f violates the Intermediate Value Theorem.

Theorem 1.5. *Suppose that X is a connected space and that $f : X \to Y$ is a continuous surjective function. Then Y is also a connected space.*

Proof. If $\{A, B\}$ is a disconnection of Y, then $\{f^{-1}(A), f^{-1}(B)\}$ is a disconnection of X. □

Corollary 1.6. *Suppose that $X \equiv Y$. Then X is connected if and only if Y is connected.*

Proof. Homeomorphisms are continuous and surjective. □

1.1. Path Connectedness. Sometimes connectedness can be a slightly awkward property to deal with. It is quite common in certain areas of mathematics to use a slightly stronger property. This property also has the added feature that it corresponds more closely to our intuitive notion of what connectedness should mean.

DEFINITION 1.7. A space X is said to be path connected if, for all $x \in X$ and $y \in X$, there is a continuous function $g : [0, 1] \to X$ such that $g(0) = x$ and $g(1) = y$.

PROPOSITION 1.8. *If X is path connected, then X is connected.*

PROOF. □

However, there are spaces that are connected but not path connected.

EXAMPLE 1.9. Let S be the following subspace of $\mathbb{R}^2$ (with the usual topology).

$$S = \{t, \sin(\frac{1}{t}) : t > 0\} \cup \{(0, y) : -1 \leq y \leq 1\}$$

This space is known as the (closed) topologist's sine curve. It is connected but not path connected.

2. Compactness

In set theory, an important property that may be possessed by a set is that of being finite. Many combinatorial arguments rest on the finiteness of a certain set and the theory of finite sets is much simpler than that of infinite sets. The topological analogue of finiteness is compactness.

Let $(X, \mathcal{T})$ be a topological space. A collection of subset of subsets of X is said cover X if its union is X - in that case the collection is called a cover of the set X. A subcover of a cover of X is a subcollection that also covers X. An open cover of X is a collection of open subsets that covers X.

DEFINITION 2.1. A topological space $(X, \mathcal{T})$ is compact if every open cover of X has a finite subcover.

EXAMPLE 2.2. Let $\mathbb{R}$ be the space of real numbers with the usual topology. The collection $\{(-n, n) : n \in \mathbb{N}\}$ is an example of an open cover of $\mathbb{R}$. Clearly this open cover has no finite subcover. Thus $\mathbb{R}$ is not compact.

EXAMPLE 2.3. Let X be a set and let $\mathcal{T}_{fc}$ be the finite complement topology on X. Then $(X, \mathcal{T}_{fc})$ is a compact topological space.

PROPOSITION 2.4. *If $f : X \to Y$ is a continuous surjective function and X is compact, then Y is also compact.*

PROOF. Let $\mathcal{O}$ be an open cover of Y and define

$$\mathcal{Q}\{f^{-1}(U) : U \in \mathcal{O}\}.$$

It is clear that $\mathcal{Q}$ is an open cover of X. So by assumption, it must have a finite subcover. That is, there are open set $U_i \in \mathcal{O}$, $i = 1, \dots, k$ such that

$$X = f^{-1}(U_1) \cup \cdots \cup f^{-1}(U_k).$$

Now since f is surjective $f(f^{-1}(A)) = A$, for any $A \subset Y$. Therefore

$$\begin{aligned} Y &= f(X) \\ &= f(f^{-1}(U_1) \cup \cdots \cup f^{-1}(U_k)) \\ &= f(f^{-1}(U_1)) \cup \cdots \cup f(f^{-1}(U_k)) \\ &= U_1 \cup \cdots \cup U_k \end{aligned}$$

So $\mathcal{O}$ has a finite subcover as required. □

COROLLARY 2.5. *If $X \equiv Y$ then X is compact if and only if Y is compact.*

PROOF. Homeomorphisms are continuous and surjective. □

2.1. Review of sequences. Suppose that (x_n) is a sequence in X.

DEFINITION 2.6. We say that (x_n) converges to $x \in X$ if, for every open subset U containing x, there is a positive integer N_U such that $n > N_U \Rightarrow x_n \in U$.

If a sequence (x_n) converges to x, we write $(x_n) \to x$ or $\lim_{n\to\infty} x_n = x$.

Note that in spaces that are not metrically induced, sequences can behave in pathological ways.

EXAMPLE 2.7. Consider the topological space $(\mathbb{N}, \mathcal{T}_{fc})$ where $\mathcal{T}_fc$ is the finite complement topology let $x_n = n$. Then for all $m \in \mathbb{N}$, $(x_n) \to m$. In other words, the sequence converges to every point in the space.

In the case of metric spaces, it is easy to see that

LEMMA 2.8. *Suppose that (Y, d) is a metric space and that (y_n) is a sequence in Y. Then $(y_n) \to y$ if and only if for every $\epsilon > 0$, there is some N_ϵ such that $n > N_\epsilon \Rightarrow d(y, y_n) < \epsilon$.*

PROPOSITION 2.9. *Suppose that $f : Y \to Z$ is a continuous function and that $(y_n) \to y$ for some sequence (y_n) in Y. Then $(f(y_n)) \to f(y)$.*

PROOF. Suppose that U is an open subset containing $f(y)$. Then, by continuity, $f^{-1}(U)$ is an open subset of X that contains y. Therefore, there is some N_U such that $n > N_U \Rightarrow y_n \in f^{-1}(U)$. Therefore $n > N_U \Rightarrow f(y_n) \in U$. □

For metric spaces we have the following converse to Proposition 2.9.

PROPOSITION 2.10. *Suppose that Y and Z are metric spaces and that $f : Y \to Z$ has the property that for every convergent sequence (y_n) in Y, $(f(y_n)) \to f(\lim_{n\to\infty} y_n)$. Then f is a continuous function.*

PROOF. Suppose that f is not continuous. So there is some closed subset B of Z, such that $f^{-1}(B)$ is not a closed subset of Y. Therefore, there is some $y \in Y - f^{-1}(B)$ such that every open set containing y meets $f^{-1}(B)$. Now, for each positive integer n, choose $y_n \in B(y, \frac{1}{n}) \cap f^{-1}(B)$. Clearly $(y_n) \to y$. But $(f(y_n))$ is a sequence in the closed subset B and $f(y)$ is not in B. Therefore $(f(y_n))$ cannot converge to $f(y)$ contradicting our hypothesis. □

Note that the conclusion of Proposition 2.10 is not necessarily true if we only assume that Y and Z are topological spaces.

2.2. Sequential Compactness. Recall also that a subsequence of a sequence (x_n) is a sequence (x_{n_k}) where n_k is a strictly increasing sequence of integers. So, in particular, a sequence that does not converge may have a convergent subsequence.

DEFINITION 2.11. A topological space X is said to be sequentially compact if every sequence in X has a convergent subsequence.

EXAMPLE 2.12. The open interval $(0, 1)$ (with the usual topology) is not compact since the sequence $(\frac{1}{n})$ has no convergent subsequence.

THEOREM 2.13. *(Bolzan-Weierstrass) Any closed bounded interval of real numbers is sequentially compact.*

PROOF. Let $[a, b]$ be a closed bounded interval and let (x_n) be a sequence in $[a, b]$.

We define a sequence (n_k) of natural numbers and sequences (a_k) and (b_k) of real numbers inductively as follows. Let $n_1 = 1$, $a_1 = a$ and $b_1 = b$. Now suppose that n_i, a_i and b_i have been chosen for some $i \geq 1$. Moreover, suppose inductively that $a_i \leq b_i$ and that the interval $[a_i, b_i]$ contains infinitely many terms of the sequence (x_n). Now, at least one of the intervals $[a_i, \frac{a_i+b_i}{2}]$ or $[\frac{a_i+b_i}{2}, b_i]$ must contain infinitely many terms of the sequence (x_n). Choose a_{i+1} and b_{i+1} so that $[a_i, b_i]$ is one of the aforementioned intervals that does contain infinitely many terms of (x_n). Clearly, we can choose a positive integer n_{i+1} so that $n_{i+1} > n_i$ and $x_{n_{i+1}} \in [a_{i+1}, b_{i+1}]$. Thus, we have chosen n_{i+1}, a_{i+1} and b_{i+1}. Moreover, $[a_{i+1}, b_{i+1}]$ contains infinitely many terms of the sequence (x_n).

Now, we claim that the sequence (x_{n_k}) is convergent. Notice that by construction

$$a_i \leq a_{i+1} \leq b_{i+1} \leq b_i$$

for all i. So (a_k) is a bounded increasing sequence of real numbers and is therefore convergent. Similarly, (b_k) is a bounded decreasing sequence of real numbers and is therefore also convergent. Moreover, for each i, $b_{i+1} - a_{i+1} = \frac{b_i - a_i}{2}$. Therefore $b_i - a_i = \frac{b-a}{2^i} \to 0$ as $i \to \infty$. Thus $\lim(a_k) = \lim(b_k)$. Moreover, for each i, $a_i \leq x_{n_i} < b_i$. So clearly, (x_{n_k}) must be convergent as a sequence in $\mathbb{R}$. Let $x = \lim(x_{n_k})$. Since $[a, b]$ is a closed subset of $\mathbb{R}$, $x \in [a, b]$. Therefore (x_{n_k}) must be convergent as a sequence in $[a, b]$. □

We can easily generalise this to higher dimensions. A subset A of $\mathbb{R}^m$ is said to be bounded if there is some M such that $d(0, a) \leq M$ for all $a \in A$.

Corollary 2.14. *A subspace of $\mathbb{R}^m$ is sequentially compact if and only if it is a closed and bounded subset of $\mathbb{R}^n$.*

Proof. Let A be a closed bounded subset of $\mathbb{R}^n$ and let (x_n) be a sequence in A. Suppose that for each n, $x_n = (x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(m)})$, where $x_n^{(i)} \in \mathbb{R}$. Then $(x_n^{(1)})$ is a bounded sequence of real numbers and thus, by Theorem 2.13, it has a convergent subsequence $(x_{n_k}^{(1)})$. Now, $(x_{n_k}^{(2)})$ is a bounded sequence of real numbers and so, by Theorem 2.13, it has a convergent subsequence $(x_{n_{k_l}}^{(1)})$. Continuing in this way we eventually construct a sequence N_k such that $(x_{N_k}^{(i)})$ is convergent for all i. Therefore (x_{N_k}) is a convergent subsequence of the original sequence. Since A is closed, it contains all its limit points, so (x_{N_k}) converges as a sequence in A.

Conversely suppose that A is not bounded, then, for every positive integer n, we can find $x_n \in A$ such that $d(0, x_n) > n$. Clearly, (x_n) has no convergent subsequence. If A is not closed then we can find a sequence (x_n) in A that converges (in $\mathbb{R}^n$) to a point outside A. As a sequence in A, (x_n) has no convergent subsequence. □

2.3. Compactness versus Sequential Compactness. In general there is no relationship between compactness and sequential compactness. There are spaces that are compact but not sequentially compact and there are spaces that are sequentially compact but not compact. However, we will focus especially on the case of metric spaces and in this case we can show that in fact the two properties are equivalent.

Proposition 2.15. *If Y is a compact metric space then it is sequentially compact.*

Proof. Suppose that (y_n) be a sequence in Y that has no convergent subsequence. For $k = 1, 2, \ldots$, let

$$G_n = Y - \{y_n, y_{n+1}, y_{n+2}, \ldots\}.$$

First, we observe that each G_k is in fact an open subset of Y. To see this, suppose that $\{y_n, y_{n+1}, y_{n+2}, \ldots\}$ is not closed. Then by Proposition 5.4, we can find a sequence (z_m) in $\{y_n, y_{n+1}, y_{n+2}, \ldots\}$ that converges to a point outside $\{y_n, y_{n+1}, y_{n+2}, \ldots\}$. Clearly this implies the existence of a convergent subsequence of (y_n) contradicting our assumption. So $\{y_n, y_{n+1}, y_{n+2}, \ldots\}$ is closed and G_n is thus an open subset of Y. Now observe that $G_i \subset G_{i+1}$ for all i. So

$$G_1 \subset G_2 \subset G_3 \subset \ldots.$$

Now, we claim that $\bigcup_{i=1}^{\infty} G_i = Y$. Suppose not, then there is some $y \in Y$ such that $y \notin G_n$ for all n. That means that $y \in \{y_n, y_{n+1}, \ldots\}$ for all n. Therefore, y occurs infinitely often in the sequence (y_n) which contradicts the assumption that (y_n) has no convergent subsequence. Therefore $\{G_n : n = 1, 2, \ldots\}$ is an open cover of the compact space Y and therefore has a finite subcover. But since

$$G_1 \subset G_2 \subset G_3 \subset \ldots,$$

that means that $G_l = Y$ for some l which is clearly false. Therefore our assumption must be false and we have shown that Y is indeed sequentially compact. □

The proof of the converse of Proposition 2.15 is a little bit trickier.

Lemma 2.16. *Suppose that $\mathcal{O}$ is an open cover of a sequentially compact metric space, then there is some strictly positive real number r such that for every $x \in X$, $B(x, r)$ is contained in some open set belonging to $\mathcal{O}$.*

Proof. Suppose the conclusion is false. Then for each positive integer n, we can choose $x_n \in X$ such that $B(x_n, \frac{1}{n})$ is not contained in any open set in $\mathcal{O}$. By assumption (x_n) has a subsequence (x_{n_k}) that converges to some x. Now $x \in G$ where $G \in \mathcal{O}$. So there is some $\epsilon > 0$ such that $B(x, \epsilon) \subset G$. Now choose k so that $d(x_{n_k}, x) < \frac{\epsilon}{2}$ and so that $\frac{1}{n_k} < \frac{\epsilon}{2}$. If $z \in B(x_{n_k}, \frac{1}{n_k})$, then

$$\begin{aligned} d(x,z) &\leq d(x, x_{n_k}) + d(x_{n_k}, z) \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Therefore $B(x_{n_k}, \frac{1}{n_k}) \subset B(x, \epsilon) \subset G$ which contradicts our choice of x_{n_k}.

□

LEMMA 2.17. *Let X be a sequentially compact metric space and let ϵ be any positive real number. There are finitely many points $x_1, \ldots, x_k$ such that $X = B(x_1, \epsilon) \cup \cdots \cup B(x_k, \epsilon)$.*

PROOF. Suppose that the conclusion is false. We will construct a sequence with no convergent subsequence, which contradicts our hypothesis. Choose $y_1 \in X$. By assumption $X \neq B(y_1, \epsilon)$ so choose $y_2 \in X - B(y_1, \epsilon)$. Now suppose that we have chosen $y_1, \ldots, y_n$. By assumption $X - B(y_1, \epsilon) \cup \cdots \cup B(y_n, \epsilon)$ is not empty, so choose $y_{n+1} \in X - B(y_1, \epsilon) \cup \cdots \cup B(y_n, \epsilon)$.

Now, we claim that the sequence (y_n) constructed above has no convergent subsequence. Observe that if $m > n$, then $y_m \notin B(y_n, \epsilon)$. Thus for all m and n, $d(y_m, y_n) > \epsilon$. Clearly, no subsequence of a sequence with this property can converge. □

PROPOSITION 2.18. *If X is a sequentially compact metric space then X is compact.*

PROOF. Let $\mathcal{O}$ be an open cover of X. By Lemma 2.16, we can choose $r > 0$ such that for all $x \in X$, $B(x, r)$ is contained in some element of $\mathcal{O}$. Now, by Lemma 2.17, we can find $x_1, \ldots, x_k$ such that $X = B(x_1, r) \cup \cdots \cup B(x_k, r)$. Now, for each x_i, let G_i be an element of $\mathcal{O}$ that contains $B(x_i, r)$. So

$$X = B(x_1, r) \cup \cdots \cup B(x_k, r) \subset G_1 \cup \cdots \cup G_k.$$

Therefore $\{G_1, \ldots, G_k\}$ is a finite subcover of $\mathcal{O}$. □

2.4. Problems.

(1) Let $\mathcal{O}$ be an open cover of a space X and define a function $L : X \to \mathbb{R}$ as follows.

$$L(x) = \min\{1, \sup\{r \in \mathbb{R} : B(x, r) \subset G, G \in \mathcal{O}\}\}.$$

(a) Show that $L(x) > 0$ for all $x \in X$.
(b) Show that L is a continuous function.

(c) Use these facts, together with the fact that all continuous real valued functions on sequentially compact spaces are bounded, to provide another proof of Lemma 2.16.