Nilpotent Spaces of Matrices and a Theorem of Gerstenhaber

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Nilpotent spaces

An $n \times n$ matrix A is *nilpotent* of degree k if $N^k = 0$ and $N^{k-1} \neq 0$.

$M_n(\mathbb{F})$ is the $\mathbb{F}$-vector space of $n \times n$ matrices with entries in $\mathbb{F}$. A *nilpotent subspace* of $M_n(\mathbb{F})$ is a linear subspace all of whose elements are nilpotent.

Given n and $\mathbb{F}$

1. What is the maximum dimension of such a subspace?
2. What are the maximal nilpotent subspaces with respect to inclusion?

To what extent do the answers to these questions depend on $\mathbb{F}$?

Examples

1. SU_n, the space of strict upper triangular matrices, has dimension $\frac{n(n-1)}{2}$.
2. There are non triangularisable examples. (Mathes, Omladič and Radjavi 1991)
$$\begin{pmatrix} 0 & a & 0 \\ -b & 0 & a \\ 0 & b & 0 \end{pmatrix}$$
There is no proper subspace of $\mathbb{F}^3$ that is invariant under all of these matrices.
3. What is the maximum dimension of an irreducible nilpotent subspace of M_n? If $n = 3m$ Mathes et al. have an example of dimension $m^2 + 1$.

A 8 dimensional irreducible nilpotent subspace of M_6

$$\begin{pmatrix} 0 & a & 0 & 0 & 0 & 0 \\ -b & 0 & a & g & c & 0 \\ 0 & b & 0 & h & d & 0 \\ 0 & 0 & 0 & 0 & a & 0 \\ -e & 0 & 0 & -f & 0 & a \\ 0 & e & 0 & 0 & f & 0 \end{pmatrix}$$

More generally, for $n = 3m$ we can construct an irreducible nilpotent subspace of dimension $\frac{5}{2}m^2 - \frac{3}{2}m + 1$.

Gerstenhaber's Theorem

Theorem

(Gerstenhaber 1958) If $\mathbb{F}$ has at least n elements then

1. *The maximal dimension of a nilpotent subspace of $M_n(\mathbb{F})$ is $\frac{n(n-1)}{2}$.*
2. *If N is a nilpotent subspace of $M_n(\mathbb{F})$ of dimension $\frac{n(n-1)}{2}$ then there is some invertible matrix E such that $E^{-1}NE = SU_n$.*

Theorem

(Serezhkin 1985) No restriction on the size of $\mathbb{F}$ is required.

Several proofs of Gerstenhaber's result have appeared in the literature - Mathes et al. 1991, Brualdi and Chavey 1993.

The Hyperplane Annihilation Property

For $A, B \in M_n$ define $\langle A, B \rangle = tr(AB)$.
$\langle , \rangle$ is a nondegenerate symmetric bilinear form on M_n.
Which subspaces of M_n are dual to nilpotent subspaces?

Definition

A linear subspace, X, of M_n has the *hyperplane annihilation property*, if, given any linear hyperplane $H \leq \mathbb{F}^n$, there is some $A \in X$ such that $tr(A) = 1$ and $Av = 0$ for all $v \in H$.

Theorem

(RQ) A subspace X of M_n has the hyperplane annihilation property if and only if every element of $X^\perp$ has no non zero eigenvalues in $\mathbb{F}$.

Corollary

If N is a nilpotent subspace of M_n then $N^\perp$ has the hyperplane annihilation property.

The converse is true over an algebraically closed field.
However, if $\mathbb{F} = \mathbb{R}$ and S is the space of symmetric matrices then

1. S has the hyperplane annihilation property - $tr(uu^T) \neq 0$ for any non zero column vector u.
2. $S^\perp$ is the space of skew symmetric matrices. So $S^\perp$ is not nilpotent.

Dualised version of Gerstenhaber's Theorem

Theorem

(RQ 2011) Suppose that E is a linear subspace of M_n that has the hyperplane annihilation property. Then $\dim(E) \geq \frac{n(n+1)}{2}$.

Theorem

(Proof RQ, JC) Suppose that E is a linear subspace of M_n that has the hyperplane annihilation property. If $\dim E = \frac{n(n+1)}{2}$ *and if $E^{\perp}$ is a nilpotent space then E is conjugate to U_n.*

Proof.

Show that there is some $v \in \mathbb{F}^n$ such that $Ev = \mathbb{F}^n$. Then use induction on n. □