An Introduction to the Rigidity Theory of Bar and Joint Frameworks

James Cruickshank

NUI Galway

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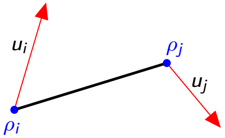
Frameworks

An n-dimensional (abstract) bar and joint framework is a pair (G, ρ) where $G = (V, E)$ is a simple graph and $\rho : V \to \mathbb{R}^n$.
Convention: $V = \{1, \cdots, k\}$ and $\rho \in \mathbb{R}^{n \times k}$.

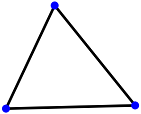
Infinitesimal flex: $u \in \mathbb{R}^{n \times k}$ such that

$$(u_i - u_j)^T (\rho_i - \rho_j) = 0 \quad \forall ij \in E$$

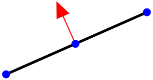


Infinitesimally rigid $\Leftrightarrow$ every flex of (G, ρ) is also a flex of (K_k, ρ) $\Leftrightarrow$ every flex is induced by some infinitesimal isometry of $\mathbb{R}^n$.

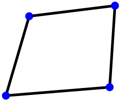
Planar Examples



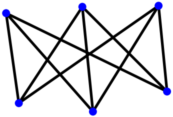
Infinitesimally rigid



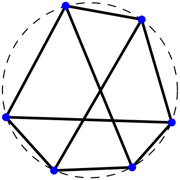
Infinitesimally flexible



Flexible



Rigid



Infinitesimally flexible

Generic Rigidity

$\rho \in \mathbb{R}^{n \times k}$ is generic if the multiset of its entries is algebraically independent over $\mathbb{Q}$.

- Generic $\Rightarrow$ general position. Generic $\Rightarrow$ every square submatrix has maximal rank.
- For generic frameworks rigidity $=$ infinitesimal rigidity.
- For generic ρ and ρ', (G, ρ) is rigid if and only if (G, ρ') is rigid. Thus *generic rigidity is a property of the graph*.

Isostatic $=$ minimally rigid with respect to the edge set.

Theorem

(Laman 1970) A graph G is generically isostatic in the plane if and only if

- $|E| = 2|V| - 3$ *(enough edges to brace the framework)*
- *For all* $V' \subset V$, $|E'| \leq 2|V'| - 3$ *(no subframework is overbraced)*

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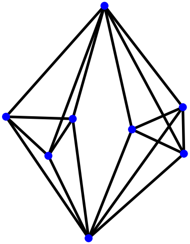
Bananas and all that

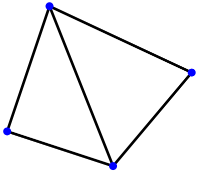


No Laman type theorem for 3 dimensional generic rigidity is known!
Double banana satisfies $|E| = 3|V| - 6$ and $|E'| \leq 3|V'| - 6$ for all $V' \subset V$.

G a 3-connected, planar graph then G is generically 3-rigid iff it satisfies the Laman counts. Proof: Cauchy rigidity theorem + Steinitz Theorem on polyhedral graphs

Henneberg Extensions

n dimensional k-extension: remove k edges and adjoin a new vertex of degree $n + k$ adjacent to all the vertices of the removed edges.



Planar 0-extension

$n = 2, k = 0$

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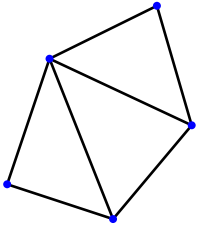

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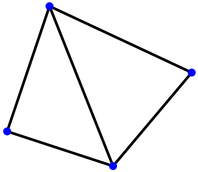

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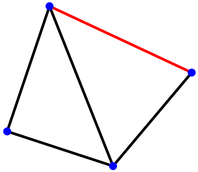

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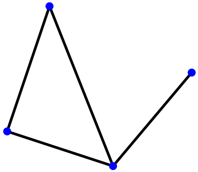

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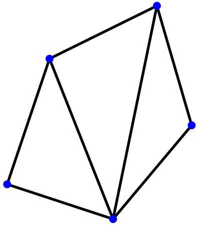

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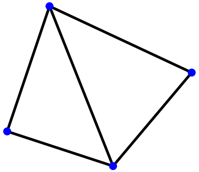

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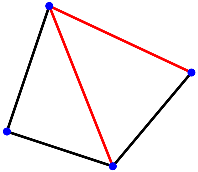

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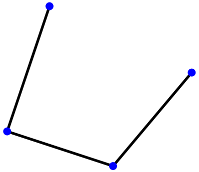

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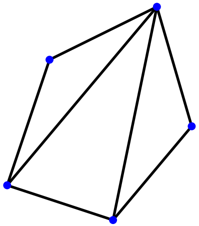

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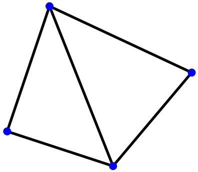

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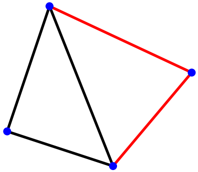

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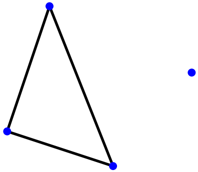

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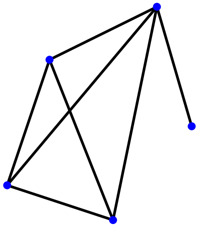

Planar 2-extension

Some 2-extensions break rigidity!

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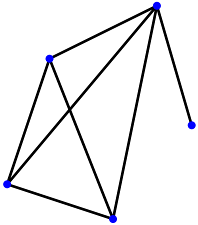

Planar 2-extension

Some 2-extensions break rigidity!

Geometric part of Laman's theorem: 2-dimensional k-extensions preserve generic rigidity for $k = 0, 1$. (easy enough to see this)

3 dimensional k-extensions preserve generic rigidity for $k = 0, 1$, but we need to know about 2-extensions in this case...