# On Spaces of Motions of Five Points in $\mathbb{R}^3$

James Cruickshank

NUI Galway

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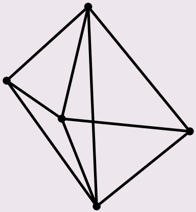
## Conjecture

*(Graver, Tay, Whiteley) Suppose that $G$ is generically 3-rigid and that suppose that some 2-extension, supported on a vertex set $X$ , and on edges $e, f$ , is not generically 3-rigid. Then $X$ is not free in $G - \{e, f\}$.*

## Example



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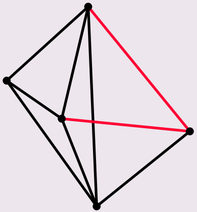

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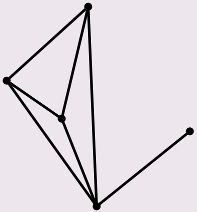

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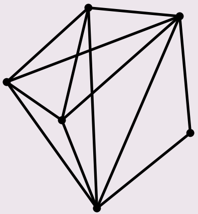

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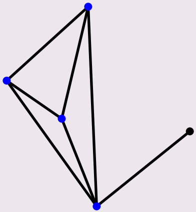
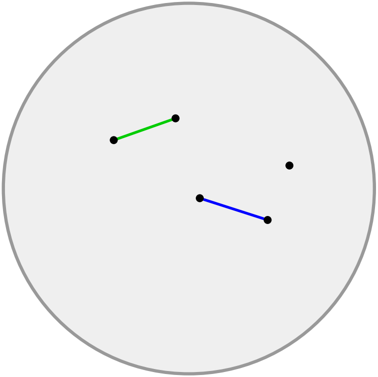


Every infinitesimal flex of this framework restricts to an isometry of the blue vertices - ie the vertex set is not free.

$(G,\rho)$ is a generic isostatic framework. $e,f \in E(G)$ and $V(\{e,f\}) \subset X \subset V$, $|X| = 5$. Then $(G - \{e,f\},\rho)$ has a two dimensional linear space of infinitesimal flexes that are all non-trivial when restricted to $X$.



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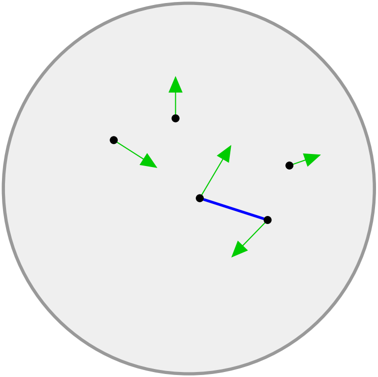

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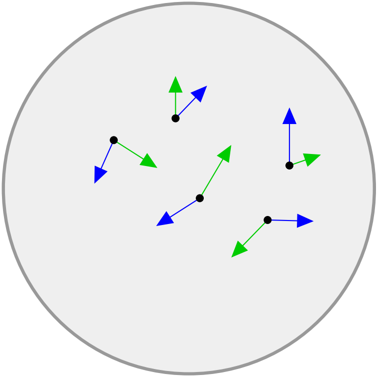
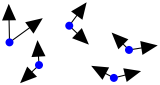


Let $p$ be a point configuration in $\mathbb{R}^3$ and $S$ a linear space of infinitesimal motions of $p$ such that $u \in S$, $u \neq 0 \Rightarrow u$ not an isometry of $p$.

**Definition**

$S$ is $p$-admissible if for almost all $x \in \mathbb{R}^3$ the framework $(K_{|p|,1}, p \cup x)$ admits a flex that restricts to a nonzero element of $S$.



Goal: Understand two dimensional $p$-admissible subspaces of $\mathbb{R}^{3\times 5}$ for generic $p \in \mathbb{R}^{3\times 5}$. Some progress on this.

Difficult question: Which $p$-admissible spaces arise from deletion of two edges from a generic isostatic framework? Only partial results in a special case.

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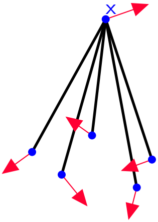


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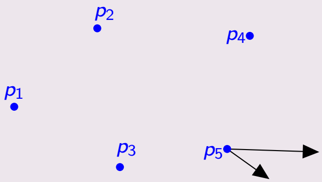
# Examples in $\mathbb{R}^3$

## Example

$p$ any 5 point configuration and

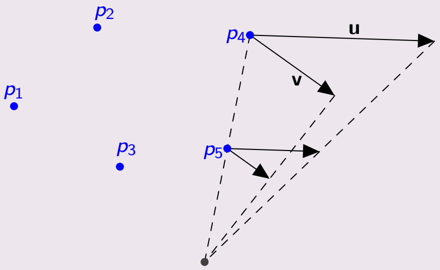
$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$




This is the $p$-admissible space that arises from the example on the first slide.

## Example

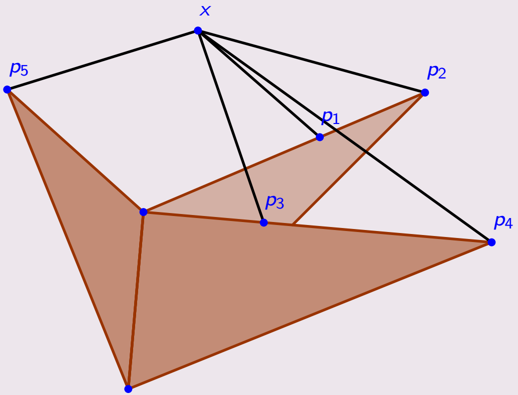
$$(p_5 - p_4)^\perp = \langle \mathbf{u}, \mathbf{v} \rangle,\ S = \begin{pmatrix} 0 & 0 & 0 & k(s\mathbf{u} + t\mathbf{v}) & s\mathbf{u} + t\mathbf{v} \end{pmatrix}$$




Both these examples consist entirely of rank one matrices. In both cases, there is a 3-set of points such that every element of $S$ restricts to an isometry on that set.

## Example

A non generic example (with 4 coplanar points).



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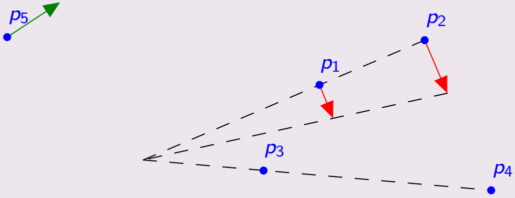


# Results

$p$ is a generic 5 point configuration in $\mathbb{R}^3$.

**Theorem**

*There is a two dimensional p-admissible space of infinitesimal motions, $S$, of p such that for every 3-point subconfiguration q there is some element of $S$ that is not an isometry of $q$.*

**Theorem**

*If $S$ is a two dimensional p-admissible space of infinitesimal motions of p then either*

- *Every element of $S$ is the restriction to p of an affine linear mapping $\mathbb{R}^3 \to \mathbb{R}^3$, or*
- *$S$ is equivalent to a space $S'$ such that every element of $S'$ has rank one.*

These suggest the following questions (neither of which I can answer).

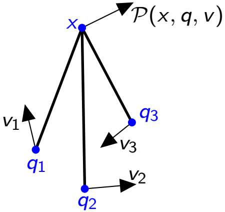
**Question**

*Given a generic isostatic framework, delete two edges. Can this give rise to an 'affine' space of infinitesimal motions of 5 points of the framework?*

**Question**

*Which rank one p-admissible spaces arise from the deletion of 2 edges from a generic isostatic framework?*

# Outline of Proofs




$q \in \mathbb{R}^{3\times 3}$ is a point configuration and $v \in \mathbb{R}^{3\times 3}$ is an infinitesimal motion of $q$ - $q_i$ resp. $v_i$ is the $i$th column of $q$ resp. $v$. $\mathbb{1}$ is the column vector all of whose entries are 1.

$$
\begin{aligned}
\mathcal{P}(x,q,v) &= \left(\mathbb{1}x^T - q^T\right)^{-1}\left(v^T x - \begin{pmatrix} v_1^T q_1 \\ v_2^T q_2 \\ v_3^T q_3 \end{pmatrix}\right) \\
&= -(q^T)^{-1}\left(I + \frac{\mathbb{1}(q^{-1}x)^T}{1-(q^{-1}x)^T\mathbb{1}}\right)\left(v^T x - \triangle(v^T q)\right)
\end{aligned}
$$

This is essentially the Sherman-Morrison-Woodbury formula.

- $K_{5,1}$ is a union of two copies of $K_{3,1}$ that have one edge in common. So for $p \in \mathbb{R}^{3\times 5}$, let $q = (p_1, p_2, p_3)$, $r = (p_1, p_4, p_5)$. For $u \in \mathbb{R}^{3\times 5}$, let $v = (u_1, u_2, u_3)$ and $w = (u_1, u_4, u_5)$.
- $S \leq \mathbb{R}^{3\times 5}$ is $p$-admissible if and only if, for all $x \in \mathbb{R}^3$, there is a non zero $u \in S$ such that

$$\mathcal{P}(x, q, v) - \mathcal{P}(x, r, w) = 0. \tag{1}$$

- If $vq^{-1} = wr^{-1}$ and $(q^T)^{-1}\triangle(v^T q) = (r^T)^{-1}\triangle(w^T r)$, then the LHS of (1) is a rank one mapping $S \to \mathbb{R}^3$.
  - $vq^{-1} = wr^{-1}$ - 9 dimensional space of solutions.
  - $(q^T)^{-1}\triangle(v^T q) = (r^T)^{-1}\triangle(w^T r)$ - at most 2 independent linear constraints - 7 dimensions left.
  - Isometries of $p$ form a 3 dimensional subspace

  Therefore there are lots of two dimensional $p$-admissible spaces ...

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…some of which do not restrict to isometries on any three points of $p$. Hence

**Theorem**

*There is a two dimensional $p$-admissible space of infinitesimal motions, $S$, of $p$ such that for every 3-point subconfiguration $q$ there is some element of $S$ that is not an isometry of $q$.*

# Necessary Conditions for admissibility

Idea (vaguely inspired by coarse geometry): Look at $\mathcal{P}(x, q, v)$ from far away.

$$\begin{aligned}\mathcal{L}(x, q, v) &:= \lim_{t \to +\infty} \frac{\mathcal{P}(tx, q, v)}{t} \\ &= -(q^T)^{-1} \left( I - \frac{\mathbb{1}(q^{-1}x)^T}{(q^{-1}x)^T \mathbb{1}} \right) v^T x\end{aligned}$$

- $I - \frac{\mathbb{1}(q^{-1}x)^T}{(q^{-1}x^T)\mathbb{1}}$ is projection onto $(q^{-1}x)^{\perp}$ with kernel $\mathbb{1}$.
- $\mathcal{L}(x, q, v)$ is a 1-homogeneous function of $x$.
- If $S$ is a two dimensional $p$-admissible subspace of $\mathbb{R}^{3\times 5}$ then for almost all $x$

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for some non-zero $u \in S$.

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...(choose bases) ...LHS of (2) is

$$A(x) = \begin{pmatrix} L_1(x) & Q_1(x) \\ L_2(x) & Q_2(x) \end{pmatrix}$$

where $L_i$ is affine linear and $Q_i$ is affine quadratic.
$p$-admissibility $\Rightarrow det(A(x)) = 0 \Leftrightarrow L_1 Q_2 - L_2 Q_1 \equiv 0$.

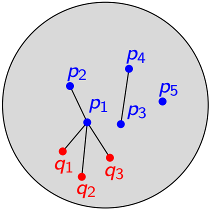
**Theorem**

*If $S$ is a two dimensional $p$-admissible space of infinitesimal motions of $p$ then either*

- *Every element of $S$ is the restriction to $p$ of an affine linear mapping $\mathbb{R}^3 \to \mathbb{R}^3$, or*
- *$S$ is equivalent to a space $S'$ such that every element of $S'$ has rank one.*

# Special Case: Vertex of Degree 4

$(G, \rho)$ generic 3-isotatic. $G'$ a 2-extension of $G$ that is not generically rigid. $e, f$ are the deleted edges and $1, 2, 3, 4, 5$ are the neighbours of the new vertex - wlog $e = (1, 2)$. $p_i = \rho_i, i = 1, \cdots, 5$. Suppose that $deg_G(1) \leq 4$. Let $S$ be the induced $p$-admissible space. Then...




## Lemma

*$S$ is equivalent to a space of rank 1 matrices.*

## Proof.

$x \mapsto \mathcal{P}(x, q, v)$ is affine linear $\Rightarrow x \mapsto \mathcal{L}(x, q, v)$ is linear.
$\therefore (q^{-1}x)^T \mathbb{1} | (q^{-1}x)^T v^T x$
$\therefore v = c\mathbb{1}^T + aq$ where $a$ is skew symmetric i.e. $v$ is an isometry of $q$ ... $\square$