MP231/209/291: 2011

Solutions to Problem Sheet Questions

These answers come from doing each question twice and checking that the results agree. However mistakes do happen, as you have often been told in the tutorials, and so if anyone does spot a mistake in these solutions please contact Neil.

SHEET 1 - Partial Differentiation

1) **(a)** $u_x = v_y = 2x$, $u_y = -v_x = -2y$. **(b)** $u_x = v_y = \frac{-x^2+y^2}{(x^2+y^2)^2}$, $u_y = -v_x = -\frac{2xy}{(x^2+y^2)^2}$.
(c) $u_x = v_y = 2e^{y^2-x^2}(y\cos(2xy) - x\sin(2xy))$, $u_y = -v_x = 2e^{y^2-x^2}(x\cos(2xy) + y\sin(2xy))$.
(d) $u_x = v_y = \frac{x}{x^2+y^2}$, $u_y = -v_x = -\frac{y}{x^2+y^2}$. **(e)** $u_x = v_y = 3x^2 - 3y^2$, $u_y = -v_x = 1 - 6xy$.
(f) $u_x = v_y = e^x\cos(y)$, $u_y = -v_x = -e^x\sin(y)$.
2) $f_{xx} = -f_{yy} = 2xy(x^2+y^2)^{-2}$.
3) **i)** $f_x = \frac{1}{2}xe^{\frac{1}{4}u}$, $f_y = -\frac{1}{2}ye^{\frac{1}{4}u}$. **(ii)** $f_x = f_u u_x = 2xf_u$, $f_y = f_u u_y = -2yf_u$.
4) **i)** $f_x = \frac{2x}{(x^2+y^2)}$, $f_y = \frac{2x}{(x^2+y^2)}$. **(ii)** $f_x = f_u u_x = 2xf_u$, $f_y = f_u u_y = 2yf_u$.
5) **(i)** $f_x = 2y(x+y)^{-2}$, $f_y = -2x(x+y)^{-2}$, $f_{xx} = -4y(x+y)^{-3}$, $f_{yy} = 4x(x+y)^{-3}$, $f_{xy} = f_{yx} = 2(x-y)(x+y)^{-3}$.
(ii) $f_x = 2y(2y-x)^{-2}$, $f_y = -2x(2y-x)^{-2}$, $f_{xx} = 4y(2y-x)^{-3}$, $f_{yy} = 8x(2y-x)^{-3}$, $f_{xy} = f_{yx} = -2(3y+x)(2y-x)^{-3}$.
6) **(a)** $f_r = e^r\big(f_x\cos(\theta) + f_y\sin(\theta)\big)$, $f_\theta = -e^r\big(f_x\sin(\theta) + f_y\cos(\theta)\big)$.
(b) $f_r = e^r\big(f_x\sec(\theta) + f_y\tan(\theta)\big)$, $f_\theta = e^r\sec(\theta)\big(f_x\tan(\theta) + f_y\sec(\theta)\big)$.
7) $x = r\cos(\theta) \Rightarrow x_r = \cos(\theta)$ & $x_\theta = -r\sin(\theta)$, $y = r\sin(\theta) \Rightarrow y_r = \sin(\theta)$ & $y_\theta = r\cos(\theta)$
$r = \sqrt{x^2+y^2} \Rightarrow r_x = \frac{x}{\sqrt{x^2+y^2}}$ & $r_y = \frac{y}{\sqrt{x^2+y^2}}$, $\theta = \arctan\left(\frac{y}{x}\right) \Rightarrow \theta_x = -\frac{y}{x^2+y^2}$ & $\theta_y = \frac{x}{x^2+y^2}$.
8) $f_x = -\frac{2xz}{(x^2+y^2)^2}$, $f_y = -\frac{2yz}{(x^2+y^2)^2}$ & $f_z = \frac{1}{x^2+y^2} = \frac{x+y}{(x^2+y^2)^2} \Rightarrow f_u = -\frac{2z}{x^2+y^2}$ & $f_v = \frac{(2xz+x+y)}{(x^2+y^2)^2}$.

SHEET 2 - Optimization

1) **(a)** $(\frac{1}{2}, \frac{1}{2})$ is a MIN where $f = -\frac{3}{4}$.
(b) There are 4 st. pts - 2 SADDLE POINTS, 1 MAX and 1 MIN and $\Delta = -36(2x-a)(2y-b)$: $(0,0)$ & $(a,b) \Rightarrow \Delta = -36ab$ and $(0,b)$ & $(a,0) \Rightarrow \Delta = 36ab$. The nature of each point depends on the signs of a and b.
(c) single st. pt. at $(-1,-2)$, $\Delta = 0 \Rightarrow$ inconclusive.
2) **(a)** $(1,1)$ is a MIN where $f = 3$.
(b) at $(0,0)$ $f_x = f_y = f_{xx} = f_{yy} = f_{xy} = \Delta = 0 \Rightarrow$ inconclusive, $(-6,6)$ is a SADDLE POINT.
(c) $(0,0)$ is a MIN where $f = 0$, $(1,1)$ is a SADDLE POINT where $f = 2e^{-2}$.
(d) $(0,\pm1)$ are SADDLE POINTS, $(-1,0)$ is a MAX and $(1,0)$ is a MIN.
(e) $(1,-1)$ is a MIN where $f = e^{-2}$.
3) **(a)** St. pts. lie on circle $C = \{(0,y,z) \in \mathbb{R}^3 \mid y^2+z^2 = 9\}$ where $f(0,y,z) = 9 \Rightarrow$ MIN.
(b) $\pm(2,4)$ are MIN where $f = 40$, $\pm\big(2\sqrt{14}, -\sqrt{14}\big)$ are MAX where $f = 90$.
(c) $\left(\frac{8}{9}, -\frac{8}{9}, \frac{4}{9}\right)$ is a MIN where $f = \frac{16}{9}$.
4) $h = \frac{r}{2}$ and $r = 5\sqrt[3]{\frac{4}{\pi}} \Rightarrow A = 150\sqrt[3]{2\pi}$.
5) Rectangle is symmetric about both axes with corners at $x = \pm\frac{1}{\sqrt{2}}$, $y = \pm\frac{1}{2\sqrt{2}}$ and Area $= 1$.
6) From origin MIN distance is $\sqrt{\frac{1}{21}}$ at $\left(\frac{2}{21}, \frac{4}{21}, \frac{1}{21}\right)$.

7) From origin MIN distance is $\frac{D}{\sqrt{A^2+B^2+C^2}}$ at $\left(\frac{AD}{A^2+B^2+C^2}, \frac{BD}{A^2+B^2+C^2}, \frac{CD}{A^2+B^2+C^2}\right)$.
8) From origin MAX distance is 3 at $(0, \pm 3, 0)$ and MIN distance is 1 at $(0, 0, \pm 1)$.
9) From origin MAX distance is $\sqrt{5}$ at $(0, \pm 2, 1)$ and MIN distance is 1 at $(\pm 1, 0, 0)$.
10) From origin MAX distance is $2\sqrt{5}$ at $\pm(2, -4)$ and MIN distance is $\sqrt{5}$ at $\pm(2, 1)$.
11) From $(1, 1, 1)$ MAX distance is $\sqrt{7 + 4\sqrt{3}}$ at $-(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$ and MIN distance is $\sqrt{7 - 4\sqrt{3}}$ at $(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$.
12 In each case the MIN value of f is 0, at $\pm(3, 0, 3)$ in **(a)** and **(b)** and at $\pm(6, 0, 6)$ in **(c)**.
13) Every point is same distance (R) from origin, so method doesn't work.

SHEET 3 - Fourier Series

1) Show that $\int_{-\pi}^{\pi} f(x)g(x)dx = 0$ in each case.

2) $f(x)$ is neither ODD nor EVEN but use $g(x) = f(x) - \frac{1}{2} = \begin{cases} \frac{1}{2}, & -3 \le x < 0 \\ -\frac{1}{2}, & 0 \le x < 3 \end{cases}$ which is ODD.
$L = 3$, $a_0 = 0$, $a_n = 0$, $b_n = \frac{(-1)^n - 1}{n\pi} \Rightarrow f(x) \simeq S(x) = \frac{1}{2} + \frac{1}{\pi}\sum_{n=0}^{n=\infty} \frac{(-1)^n - 1}{n} \sin\left(\frac{n\pi x}{3}\right)$.
NB: $(-1)^n - 1 = 0$ when n is even or -2 when n is odd

3) $f(x)$ is neither ODD nor EVEN. $L = \pi$, $a_0 = \frac{\pi^2}{3}$, $a_n = \frac{2(-1)^n}{n^2}$, $b_n = \frac{2((-1)^n - 1) - \pi^2 n^2 (-1)^n}{\pi n^3} \Rightarrow$
$f(x) \simeq S(x) = \frac{\pi^2}{6} + 2\sum_{n=0}^{n=\infty} \left(\frac{(-1)^n}{n^2} \cos(nx) + \frac{((-1)^n - 1) - \pi^2 n^2 (-1)^n}{\pi n^3} \sin(nx)\right)$.
4) $f(x)$ is EVEN so $b_n = 0$. $L = 1$, $a_0 = \frac{8}{3}$, $a_n = \frac{16(-1)^n}{n^2\pi^2} \Rightarrow f(x) \simeq S(x) = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=0}^{n=\infty} \frac{(-1)^n}{n^2} \cos(n\pi x)$.
From definition $f(0) = 0$ and $f(0) = S(0)$ as f is continuous at $x = 0$, so $S(0) = 0$.
5) $f(x)$ is neither ODD nor EVEN. $L = 2$, $a_0 = 2$, $a_n = \frac{4(1-(-1)^n)}{n^2\pi^2}$, $b_n = -\frac{4(-1)^n}{n\pi} \Rightarrow$
$f(x) \simeq S(x) = 1 + \frac{4}{\pi^2}\sum_{n=0}^{n=\infty} \left(\frac{(1-(-1)^n)}{n^2} \cos\left(\frac{n\pi x}{2}\right) - \frac{\pi(-1)^n}{n} \sin\left(\frac{n\pi x}{2}\right)\right)$. [NB: $1 - (-1)^n = 0$ when n is even or 2 when n is odd]
6) $f(x)$ is EVEN so $b_n = 0$. $L = \pi$, $a_0 = -\pi$, $a_n = \frac{2(1-(-1)^n)}{n^2\pi} \Rightarrow$
$f(x) \simeq S(x) = -\frac{\pi}{2} + \frac{2}{\pi}\sum_{n=0}^{n=\infty} \frac{(1-(-1)^n)}{n^2} \cos(nx)$.
7) $f(x)$ is neither ODD nor EVEN but use $g(x) = f(x) - \pi = x$ which is ODD. $L = \pi$, $a_0 = 0$, $a_n = 0$, $b_n = -\frac{2(-1)^n}{n\pi} \Rightarrow f(x) \simeq S(x) = \pi - \frac{2}{\pi}\sum_{n=0}^{n=\infty} \frac{(-1)^n}{n} \sin(nx)$.
8) Not done yet.
9) $f(x)$ is neither ODD nor EVEN, but can separate $f(x)$ into the sum of $g(x) = x^2$ EVEN which gives a_0 and a_n, $h(x) = x$ ODD which gives b_n and a constant 1 which is added to $\frac{a_0}{2}$. $L = \pi$, $a_0 = \frac{2\pi^2}{3}$, $a_n = -\frac{4(-1)^n}{n^2}$, $b_n = -\frac{2(-1)^n}{n} \Rightarrow f(x) \simeq S(x) = 1 + \frac{\pi^2}{3} - 2\sum_{n=0}^{n=\infty} \frac{2(-1)^n}{n^2}\left(2\cos(nx) + n\sin(nx)\right)$.
10) Not done yet.
11) $f(x)$ is EVEN $\Rightarrow b_n = 0$. $L = \pi$, $a_0 = \frac{2}{\alpha\pi}(e^{\alpha\pi} - 1)$, $a_n = \frac{2\alpha}{\pi(n^2+\alpha^2)}\left((-1)^n e^{\alpha\pi} - 1\right) \Rightarrow f(x) \simeq S(x) = \frac{1}{\alpha\pi}(e^{\alpha\pi} - 1) + \frac{2\alpha}{\pi}\sum_{n=0}^{n=\infty} \frac{1}{n^2+\alpha^2}\left((-1)^n e^{\alpha\pi} - 1\right) \cos(nx)$.

SHEET 4 - Double Integrals

The following notations will be used: I_{xy} and R_{xy} indicate that the integration is to be done first w.r.t. x then w.r.t. y, i.e. $I_{xy} = \int\int_{R_{xy}} f(x, y)dxdy$ & $I_{yx} = \int\int_{R_{yx}} f(x, y)dydx$.
1) **(a)** $I = (b - a)(d - c)$, **(b)** $I = e - 2$, **(c)** $I = \frac{7}{2}$, **(d)** $I = \frac{2-\sqrt{2}}{\pi}$.
2) **(a)** $R_{xy} = \{0 \le x \le y,\ 0 \le y \le 1\} \Rightarrow R_{yx} = \{x \le y \le 1,\ 0 \le x \le 1\}$. $I = \frac{1}{2}$.
(b) $R_{xy} = \{y \le x \le 1,\ 0 \le y \le 1\} \Rightarrow R_{yx} = \{0 \le y \le x,\ 0 \le x \le 1\}$. $I = \frac{1}{10}$.
(c) $R_{xy} = \{-y \le x \le y,\ 0 \le y \le 1\} \Rightarrow I_{yx}$ is the sum of two separate integrations with $R1_{yx} = \{-x \le y \le 1,\ -1 \le x \le 0\}$ & $R2_{yx} = \{x \le y \le 1,\ 0 \le x \le 1\}$. $I = \frac{2}{3}$.

(d) $R_{yx} = \{0 \leq y \leq x,\ 0 \leq x \leq 1\} \Rightarrow R_{xy} = \{y \leq x \leq 1,\ 0 \leq y \leq 1\}$. $I = \frac{1}{15}$.
3) (a) $R_{yx} = \{0 \leq y \leq \sqrt{1-x^2},\ 0 \leq x \leq 1\} \Rightarrow R_{xy} = \{0 \leq x \leq \sqrt{1-y^2},\ 0 \leq y \leq 1\}$. $I = \frac{1}{3}$.
(b) $R_{yx} = \{0 \leq y \leq x^2,\ 0 \leq x \leq 1\} \Rightarrow R_{xy} = \{0 \leq x \leq \sqrt{y},\ 0 \leq y \leq 4\}$. $I = 4$.
(c) $R_{xy} = \{0 \leq x \leq \sqrt{1-y^2},\ 0 \leq x \leq 2\} \Rightarrow R_{yx} = \{0 \leq y \leq \sqrt{1-x^2},\ 0 \leq x \leq 1\}$. $I = \frac{7}{8}$.
(d) $R_{xy} = \{-\frac{y}{2} \leq x \leq y,\ 0 \leq x \leq 1\} \Rightarrow I_{yx}$ is the sum of two separate integrations with $R1_{yx} = \{-2x \leq y \leq 2,\ -1 \leq x \leq 0\}$ & $R2_{yx} = \{x \leq y \leq 2,\ 0 \leq x \leq 2\}$. $I = \frac{3}{2} - \cos 4 - \cos 2$.
4) (a) Cannot be integrated in original order. R is a simple rectangle so order can be switched without adjustment. $I = 2 - \sqrt{2}$.
(b) Easier in original order. $R_{yx} = \{1 + \frac{x}{2} \leq y \leq 2,\ 0 \leq x \leq 2\}$. $I = \frac{2}{3}$.
(c) Cannot(?) be integrated in original order. $R_{yx} = \{\frac{1}{2}\sqrt{x} \leq y \leq \sqrt{x},\ 0 \leq y \leq \infty\}$. $I = \frac{3}{2}$.
(d) Cannot be integrated in original order. $R_{yx} = \{x \leq y \leq 1,\ 0 \leq x \leq 1\}$. $I = \frac{1}{2}(3-e)$
(e) and **(f)** Not done yet.
5) Intersections at $(0,1)$ and $(2,5)$, $R_{yx} = \{x^2+1 \leq y \leq 2x+1,\ 0 \leq x \leq 2\} \Rightarrow A = \frac{4}{3}$.
6) Intersections at $(\pm\frac{4}{\sqrt{5}}, \frac{4}{5})$, $R_{yx} = \{4 - x^2 \leq y \leq 2x+1,\ -\frac{4}{\sqrt{5}} \leq x \leq \frac{4}{\sqrt{5}}\} \Rightarrow A = \frac{512}{9\sqrt{5}}$.
7) (a) $R_{xy} = \{0 \leq x \leq y,\ 0 \leq y \leq 1\}$. $I = \frac{1}{2}(e-1)$.
(b) I_{xy} is the sum of two separate integrations with $R1_{xy} = \{0 \leq x \leq y,\ 0 \leq y \leq 1\}$, $R2_{xy} = \{0 \leq x \leq 2-y,\ 1 \leq y \leq 2\}$. $I = \frac{9}{8} + \ln 4$.
(c) $R_{xy} = \{0 \leq x \leq \sqrt{y},\ 0 \leq y \leq 1\}$. $I = \frac{\sqrt{2}-1}{3}$.
8) $R_{r\theta} = \{\frac{1}{2} \leq r \leq 2,\ \pi \leq \theta \leq \frac{3\pi}{2}\}$. **(a)** $f = \frac{1}{r}$, $I = \frac{3\pi}{4}$, **(b)** $f = r^3\theta$, $I = \frac{1275\pi^2}{8192}$.
9) $R_{r\theta} = \{0 \leq r \leq 2,\ \frac{3\pi}{2} \leq \theta \leq 2\pi\}$, $f = 2r\big(\cos(\theta) + \sin(\theta)\big)$. $I = 0$.
10) (a) $R_{r\theta} = \{0 \leq r \leq a,\ 0 \leq \theta \leq 2\pi\}$, $f = r^3\big(\cos(\theta)^2\sin(\theta) + \cos(\theta)\sin(\theta)^2\big) + 3$. $I = 6\pi a^2$.
(b) $R_{r\theta} = \{0 \leq r \leq a,\ -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}\}$, $f = r^2 + 2r\big(\cos(\theta) - \sin(\theta)\big)$. $I = \frac{\pi a^3}{6} + \sqrt{2}a^2$.
(c) $R_{r\theta} = \{1 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi\}$ $f = \frac{1}{2}re^{-r^2}$. $I = \frac{\pi(e^3-1)}{e^4}$.
(d) $R_{r\theta} = \{a \leq r \leq b,\ 0 \leq \theta \leq \frac{\pi}{2}\}$, $f = \ln(r^2)$. $I = \frac{\pi}{4}\big((b^2-1)\ln(b^2) - (a^2-1)\ln(a^2)\big)$.
11) $R_{r\theta} = \{0 \leq r \leq 1,\ 0 \leq \theta \leq \frac{\pi}{2}\}$, $f = \frac{1}{2}re^{-r^2}\sin(2\theta)$. $I = \frac{e-1}{e}$.

SHEET 5 - Green's Theorem and Line Integration

The following notations will be used;
$L\{h(x,y)=c\}_{(x_a,y_a)}^{(x_b,y_b)} = \int_{(x_a,y_a)}^{(x_b,y_b)} fdx + gdy$ along the line/curve defined by $h(x,y) = c$.
$C\{h(x,y)=c\} = \oint fdx + gdy$ around the closed curve defined by $h(x,y) = c$.
1) In all parts $g_x - f_y = 10y$.
(a) $R_{yx} = \{x^2 \leq y \leq \sqrt{x},\ 0 \leq x \leq 1\} \Rightarrow I = \frac{3}{2}$.
$L\{y = x^2\}_{(0,0)}^{(1,1)} = -1$, $L\{y = \sqrt{x}\}_{(1,1)}^{(0,0)} = \frac{5}{2}$.
(b) $R_{xy} = \{0 \leq x \leq y-1,\ 0 \leq y \leq 1\} \Rightarrow I = \frac{5}{3}$.
$L\{x = 0\}_{(0,1)}^{(0,0)} = -2$, $L\{y = 0\}_{(0,0)}^{(1,0)} = 1$, $L\{x = 1-y\}_{(1,0)}^{(0,1)} = \frac{8}{3}$.
(c) $R_{yx} = \{0 \leq y \leq 1-x^2,\ -1 \leq x \leq 1\} \Rightarrow I = \frac{16}{3}$.
$L\{y = 0\}_{(-1,0)}^{(1,0)} = 2$, $L\{y = 1-x^2\}_{(1,0)}^{(-1,0)} = \frac{10}{3}$.
2) $g_x - f_y = 2x(1-y)$, $R_{xy} = \{\frac{1}{8}y^2 \leq x \leq 2,\ -4 \leq y \leq 4\} \Rightarrow I = \frac{128}{5}$.
$L\{x = 2\}_{(-4,2)}^{(4,2)} = 24$, $L\{x = \frac{1}{8}y^2\}_{(4,2)}^{(-4,2)} = \frac{8}{5}$.
(3) $g_x - f_y = 2$, convert to eliptical polar coordinates: $x = ar\cos(\theta),\ y = br\sin(\theta) \Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} = abr$.
$R_{r,\theta} = \{0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi\} \Rightarrow I = \int\int_{R_{r,\theta}} (2)(abr)drd\theta = 2ab\pi$.
$dx = -ar\sin(\theta),\ dy = ar\cos(\theta) \Rightarrow C\{r = 1\} = \int_0^{2\pi} abd\theta = 2ab\pi$.
(4) to **(7)** not done yet.