Chapter 2

Method of Weighted Residuals

Prior to development of the Finite Element Method, there existed an approximation technique for solving differential equations called the *Method of Weighted Residuals* (MWR). This method will be presented as an introduction, before using a particular subclass of MWR, the Galerkin Method of Weighted Residuals, to derive the element equations for the finite element method.

Suppose we have a linear differential operator D acting on a function u to produce a function p.

$$D(u(x)) = p(x).$$

We wish to approximate u by a functions $\tilde{u}$, which is a linear combination of basis functions chosen from a linearly independent set. That is,

$$u \cong \tilde{u} = \sum_{i=1}^{n} a_i \varphi_i \tag{2.1}$$

Now, when substituted into the differential operator, D, the result of the operations is not, in general, $p(x)$. Hence a error or *residual* will exist:

$$E(x) = R(x) = D(\tilde{u}(x)) - p(x) \neq 0.$$

The notion in the MWR is to force the residual to zero in some average sense over the domain. That is

$$\int_X R(x)W_i dx = 0 \qquad i = 1, 2, ..., n \tag{2.2}$$

where the nuber of *weight functions* W_i is exactly equal the number of unknow constants a_i in $\tilde{u}$. The result is a set of n algebraic equations for the unknown constants a_i.There are (at least) five MWR sub-methods, according to the choices for the W_i's. These five methods are:

1. collocation method.
2. Sub-domain method.
3. Least Squares method.
4. Galerkin method.
5. Method of moments.

Each of these will be explained below. Two examples are then given illustrating their use.

2.1 Collocation Method

In this method, the weighting functions are taken from the family of Dirac δ functions in the domain. That is, $W_i(x) = \delta(x - x_i)$. The Dirac δ function has the property that

$$\delta(x - x_i) = \begin{cases} 1 & x = x_i \\ 0 & \text{otherwise} \end{cases} .$$

Hence the integration of the weighted residual statement results in the forcing of the residual to zero at specific points in the domain. That is, integration of 2.2 with $W_i(x) = \delta(x - x_i)$ results in

$$R(x_i) = 0$$

2.2 Sub-domain Method

This method doesn't use weighting factors explicity, so it is not,strictly speaking, a member of the Weighted Residuals family. However, it can be considered a modification of the collocation method. The idea is to force the weighted residual to zero not just at fixed points in the domain, but over various subsections of the domain. To accomplish this, the weight functions are set to unity, and the integral over the entire domain is broken into a number of subdomains sufficient to evaluate all unknown parameters. That is

$$\int_X R(x)W_i dx = \sum_i \left(\int_{X_i} R(x)dx \right) = 0 \qquad i = 1, 2, ..., n$$

2.3 Least Squares Method

If the continuous summation of all the squared residuals is minimized, the rationale behind the name can be seen. In other words, a minimum of

$$S = \int_X R(x)R(x)dx = \int_X R^2(x)dx.$$

In order to achieve a minimum of this scalar function, the derivatives of S with respect to all the unknown parameters must be zero. That is,

$$\begin{aligned} \frac{\partial S}{\partial a_i} &= 0 \\ &= 2\int_X R(x)\frac{\partial R}{\partial a_i}dx \end{aligned}$$

Comparing with 2.2, the weight functions are seen to be

$$W_i = 2\frac{\partial R}{\partial a_i}$$

however, the “2” can be dropped, since it cancels out in the equation. Therefore the weight functions for the Least Squares Method are just the dierivatives of the residual with respect to the unknown constants:

$$W_i = \frac{\partial R}{\partial a_i}$$

2.4 Galerkin Method

This method may be viewed as a modification of the Least Squares Method. Rather than using the derivative of the *residual* with respect to the unknown a_i, the derivative of the approximating function is used. That is, if the function is approximated as in 2.1, then the weight functions are

$$W_i = \frac{\partial \tilde{u}}{\partial a_i}$$

Note that these are then identical to the original basis functions appearing in 2.1

$$W_i = \frac{\partial \tilde{u}}{\partial a_i} = \varphi_i(x)$$

2.5 Method of Moments

In this method, the weight functions are chosen from the family of polynomials. That is

$$W_i = x^i \qquad i = 0, 1, 2, ..., n-1$$

In the event that the basis functions for the approximation (the φ_i's) were chosen as polynomial, then the method of moments may be identical to the Galerkin method.

2.6 Example

As an example, consider the solution of the following mathematical problem. Find $u(x)$ that satisfies

$$\begin{aligned} \frac{d^2u}{dx^2} + u &= 1 \\ u(0) &= 1 \\ u(1) &= 0. \end{aligned}$$

Note that for this problem the differential operator $D(u(x))$ and $p(x)$ are

$$\begin{aligned} D(u(x)) &= (\frac{d^2}{dx^2} + 1)u(x) \\ p(x) &= 1 \end{aligned}$$

For reference, the exact solution can be found and is, in general form,

$$u(x) = C_1 \sin x + C_2 \cos x + 1$$

and for the given boundary conditions the constants can be evaluated

$$\begin{aligned} u(0) = 1 &\implies C_2 = 0 \\ u(1) = 0 &\implies C_1 = -1/\sin(1) \end{aligned}$$

So the exact solution is

$$u(x) = 1 - \frac{\sin x}{\sin(1)}$$

Let's solve by the Method of Weighted Residuals using a polynomial function as a basis. That is, let the approximating function $\widetilde{u}(x)$ be

$$\widetilde{u}(x) = a_0 + a_1 x + a_2 x^2.$$

Application of the boundary conditions reveals

$$\begin{aligned} \widetilde{u}(0) &= 1 = a_0 \\ \widetilde{u}(1) &= 0 = 1 + a_1 + a_2 \end{aligned}$$

or

$$a_1 = -(1 + a_2)$$

and the approximating polynomial which also satisfies the boundary conditions is then

$$\begin{aligned} \widetilde{u}(x) &= 1 - (1 + a_2)x + a_2 x^2 \\ &= 1 - x + a_2(x^2 - x). \end{aligned}$$

To find the residual $R(x)$, we need the second derivative of this function, which is simply $d^2\widetilde{u}/dx^2 = 2a_2$. So the residual is

$$\begin{aligned} R(x) &= \frac{d^2\widetilde{u}}{dx^2} + \widetilde{u} - 1 \\ &= 2a_2 + (1 - x + a_2(x^2 - x)) - 1 \\ &= -x + a_2(x^2 - x + 2) \end{aligned}$$

2.6.1 Collocation Method

For the collocation method, the residual is forced to zero at a number of discrete points. Since there is only one unknown (a_2), only one collocation point is needed. We choose (arbitrarily, but from symmetry considerations) the collocation point $x = 0.5$. Thus, the equation needed to evaluate the unknown a_2 is

$$R(0.5) = -0.5 + a_2(0.25 - .5 + 2) = 0$$

So

$$a_2 = +0.5/1.75 = 2/7 = 0.285714$$

2.6.2 Subdomain Method

Since we have one unknown constant, we choose a single "subdomain" which covers the entire range of x. Therefore, the relation to evalutate the constant a_2 is

$$\begin{aligned}
\int_0^1 1 \cdot R(x)dx &= 0 \\
\int_0^1 \left[-x + a_2(x^2 - x + 2)\right] dx &= 0 \\
\left[-\frac{x^2}{2} + a_2(\frac{x^3}{3} - \frac{x^2}{2} + 2x)\right|_0^1 &= 0
\end{aligned}$$

So

$$a_2(\frac{1}{3} - \frac{1}{2} + 2) = \frac{1}{2}$$

and

$$a_2 = 3/11 = 0.2727\overline{27}.$$

2.6.3 Least-Squares Method

The weight function W_1 is just the derivative of $R(x)$ with respect to the unknown a_2:

$$W_1(x) = \frac{dR}{da_2} = x^2 - x + 2$$

So the weighted residual statement becomes

$$\begin{aligned} \int_0^1 W_1(x) \cdot R(x)dx &= 0 \\ \int_0^1 \left(x^2 - x + 2\right) \cdot \left[-x + a_2(x^2 - x + 2)\right] dx &= 0 \end{aligned}$$

The math is considerably more involved than before, but nothing more than integration of polynomial terms. Direct evaluation leads to the algebraic relation

$$-\frac{11}{12} + \frac{101}{30}a_2 = 0$$

So

$$a_2 = \frac{11}{12} \cdot \frac{30}{101} = 165/606 = 0.27\overline{2277}$$

2.6.4 Galerkin Method

In the Galerkin Method, the weight function W_1 is the derivative of the approximating function $\tilde{u}(x)$ with respect to the unknown coefficient a_2:

$$W_1(x) = \frac{d\tilde{u}}{da_2} = x^2 - x$$

So the weighted residual statement becomes

$$\begin{aligned} \int_0^1 W_1(x) \cdot R(x)dx &= 0 \\ \int_0^1 \left(x^2 - x\right) \cdot \left[-x + a_2(x^2 - x + 2)\right] dx &= 0 \end{aligned}$$

Again, the math is straightforward but tedious. Direct evaluation leads to the algebraic equation:

$$\frac{1}{12} - \frac{3}{10}a_2 = 0$$

So

$$a_2 = \frac{1}{12} \cdot \frac{10}{3} = 5/18 = 0.27\overline{7}$$

2.6.5 Method of Moments

Since we have only one unknown coefficient, the weight function $W_1(x)$ is simply

$$W_1(x) = x^0 = 1.$$

As a result, the method of moments degenerates into the subdomain method for this case. Hence,

$$a_2 = 3/11 = 0.2727\overline{27}.$$

2.6.6 Comparison

A table of the tabulated values resulting from the different approximations is shown in Table 2.1 below, and a graphical comparison is seen in Figure 2.1. Figure 2.2 shows the *relative* errors for each method, as a percentage of the exact solution. Note the relative errors climb near $x = 1$, but this is largely due to the function values going to zero at that location.

RMS Errors

A reasonable scalar index for the closeness of two functions is the L_2 norm, or Euclidian norm. This measure is often called the root-mean-squared (RMS) error in engineering. The RMS error can be defined as

$$E_{RMS} = \frac{\sqrt{\int \left(u(x) - \tilde{u}(x)\right)^2 dx}}{\int dx}$$

which in discrete terms can be evaluated as

$$E_{RMS} = \sqrt{\frac{\sum_{i=1}^{N} \left(u_i - \tilde{u}_i\right)^2}{N}}.$$

The RMS errors for the different approximations are shown in the last line of Table 2.1. Note that these RMS errors are all similar in magnitude, and that the Galerkin method has a slightly lower RMS error than the others.

Table 2.1: Comparison of Different Approximations in Example 1.

x	exact	collocation	Subdomain	LeastSquares	Galerkin
0.00	1.0000	1.0000	1.0000	1.0000	1.0000
0.05	0.94060	0.93643	0.93705	0.93707	0.93681
0.10	0.88136	0.87429	0.87545	0.87550	0.87500
0.15	0.82241	0.81357	0.81523	0.81528	0.81458
0.20	0.76390	0.75429	0.75636	0.75644	0.75556
0.25	0.70599	0.69643	0.69886	0.69895	0.69792
0.30	0.64881	0.64000	0.64273	0.64282	0.64167
0.35	0.59250	0.58500	0.58795	0.58806	0.58681
0.40	0.53722	0.53143	0.53455	0.53465	0.53333
0.45	0.48309	0.47929	0.48250	0.48261	0.48125
0.50	0.43025	0.42857	0.43182	0.43193	0.43056
0.55	0.37884	0.37929	0.38250	0.38261	0.38125
0.60	0.32898	0.33143	0.33455	0.33465	0.33333
0.65	0.28080	0.28500	0.28795	0.28806	0.28681
0.70	0.23441	0.24000	0.24273	0.24282	0.24167
0.75	0.18994	0.19643	0.19886	0.19895	0.19792
0.80	0.14750	0.15429	0.15636	0.15644	0.15556
0.85	0.10718	0.11357	0.11523	0.11528	0.11458
0.90	0.06910	0.07429	0.07545	0.07550	0.07500
0.95	0.03334	0.03643	0.03705	0.03707	0.03681
1.00	0.00000	0.00000	0.00000	0.00000	0.00000
	RMS Errors	0.00591	0.00584	0.00585	0.00576

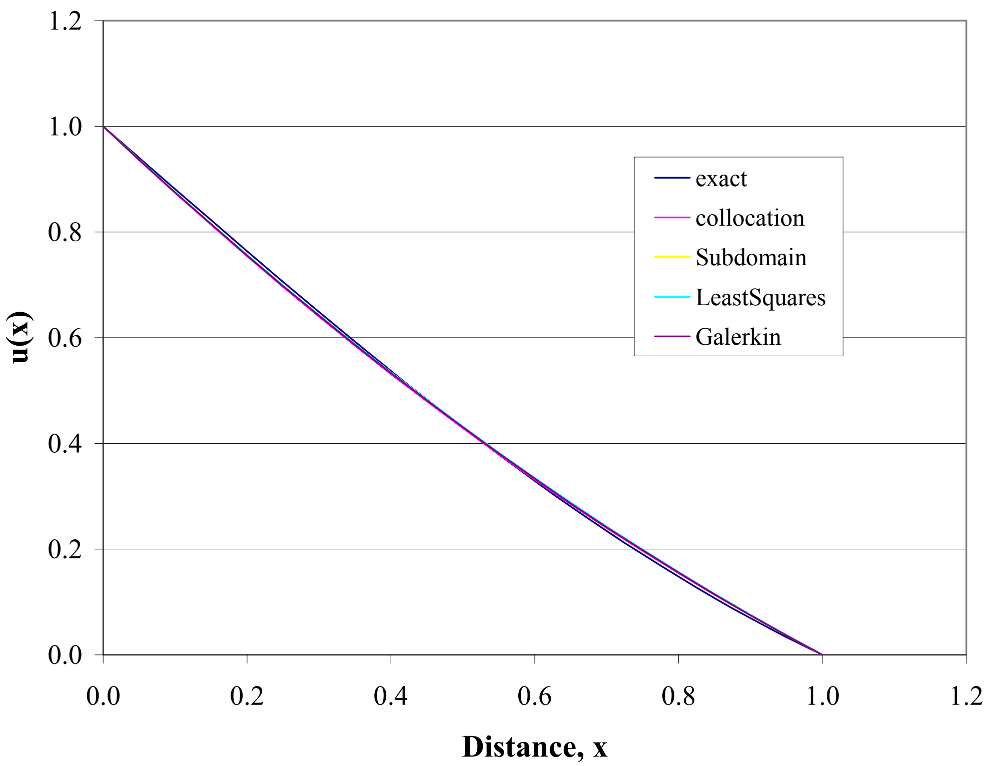

Figure 2.1: Graphical Comparison of Exact and Approximate Solutions for Example 1

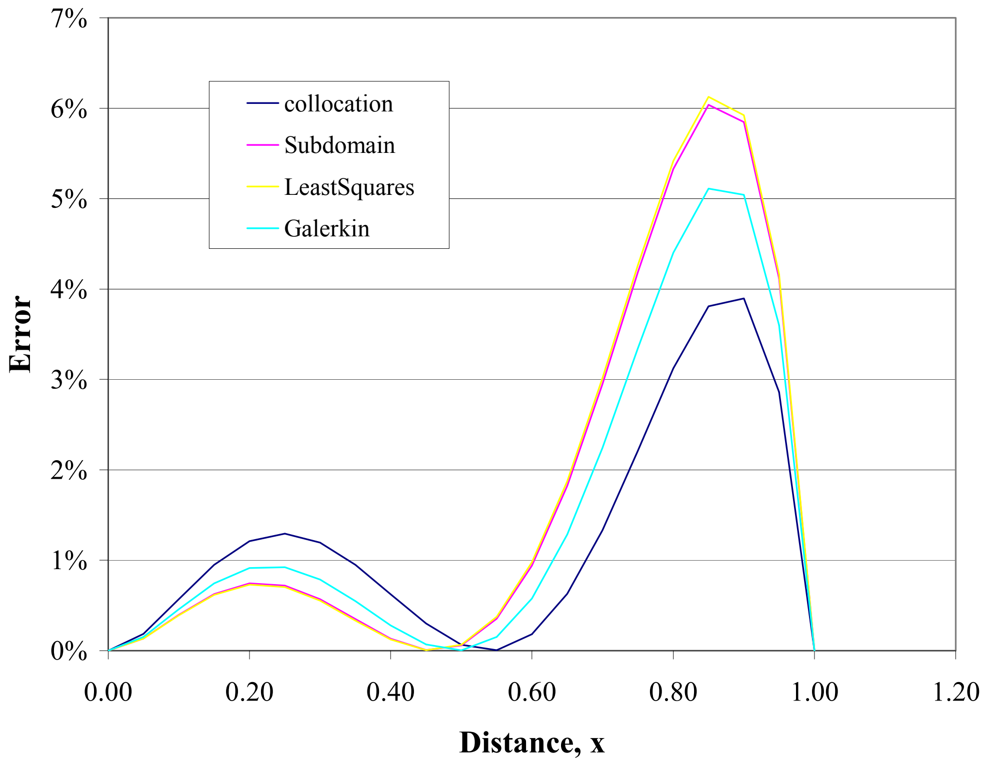

Figure 2.2: Relative Errors Between Approximate and Exact Solutions for Example 1.

2.7 References

Grandin, H., *Fundamentals of the Finite Element Method,* Waveland Press, 1991