A TUTORIAL ON THE GALERKIN METHOD

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1. Introduction

This article is a brief tutorial on the Galerkin method for solving differential equations. While it is intended to be as simple as possible, you will need to have a good background in mathematics up through and including differential equations. You should also have at least some familiarity with the basic ideas of vector spaces.

2. The Galerkin Method

For our introduction to the Galerkin method I'll follow the basic procedure as outlined in [1]. The differential equation we'll solve is

$$\frac{d}{dt}u(t) = \lambda u(t) \tag{1}$$

where $u(t)$ is the unknown function to be found and $\lambda > 0$ is a known constant. In addition we have the initial condition

$$u(0) = u_0 \tag{2}$$

with u_0 given. We seek a solution valid for $0 < t \leq 1$. This particular problem has an analytic solution, namely

$$u(t) = u_0 e^{\lambda t},$$

so there is really no need to use a numerical method to find a solution. However having an exact answer will allow us to check the validity of our numerical work.

We will seek approximate solutions to this differential equation in a vector space of functions. You may be familiar with vector spaces in 3 dimensional space in which vectors are represented as arrows, or as ordered 3-tuples of coordinates. However the concept of a vector space is much broader than this, and for our purpose here we will deal with a vector space in which the individual vectors are *functions* rather than arrows or 3-tuples.

The first step towards solving equation 1 is to choose a set of basis functions to use in representing the solution. For this case we will use the monomials $\left\{t^j\right\}_{j=0}^{q}$, where q is an integer we choose as the highest power of t in our computed solution. We will call the numerical solution $U(t)$, representing our approximation of the exact solution $u(t)$. Using these basis functions we can write the solution as

$$U(t) = u_0 + \sum_{j=1}^{q} \xi_j t^j \tag{3}$$

where the coefficients ξ_j must be found. Note that since $U(0) = u_0$, we know that $\xi_0 = u_0$ and so we pull this term out of the expression for $U(t)$ and sum from $j = 1$ rather than 0.

Date: August 6, 2009.

Step 2 is to define the *residual error function* $R(t)$ which shows the error of our computed solution. In our current example we will move all the terms of the differential equation 1 to the left-hand side of the equation and use that expression as the residual error:

$$R(U(t)) = \dot{U}(t) - \lambda U(t). \tag{4}$$

Step 3 is to apply the *Galerkin orthogonality* principle. For a general vector $v(t)$ in our vector space of functions we want to find the solution $U(t)$ such that the residual error function is orthogonal to each $v(t)$. From the theory of vector spaces, two vectors are orthogonal if their inner product is zero. By definition the inner product of two functions $w(t)$ and $v(t)$ in our vector space of functions is given by

$$\int_0^1 w(t)v(t)\,dt,$$

so the orthogonality condition becomes

$$\int_0^1 R(t)v(t)\,dt = 0$$

or

$$\int_0^1 \left(\dot{U(t)} - \lambda U(t)\right) v(t)\,dt = 0. \tag{5}$$

Step 4 is to substitute the expression for the unknown function into the orthogonality condition. Substituting equation 3 into equation 5 gives us

$$\int_0^1 \left(j\sum_{j=1}^{q} \xi_j t^{j-1} - \lambda u_0 - \lambda \sum_{j=1}^{q} \xi_j t^j \right) v(t)\,dt = 0 \tag{6}$$

which must hold for all $v(t)$. It is sufficient to have this equation be true for $v(t)$ equal to the basis functions $\left\{t^i\right\}_{i=1}^q$, enabling us to write this equation as

$$\sum_{j=1}^{q} j\xi_j \int_0^1 t^{i+j-1}\,dt - \lambda \sum_{j=1}^{q} \xi_j \int_0^1 t^{i+j}\,dt = \lambda u_0 \int_0^1 t^i\,dt \tag{7}$$

where we now have a *set* of equations, one for each i in $1, 2, \ldots, q$.

Step 5 is to solve this system of equations. We have a set of q equations in q unknowns ξ_j, so we should be able to find a unique solution. In equation 7 the integrals are easy to calculate, and we end up with

$$\sum_{j=1}^{q} \left(\frac{j}{i+j} - \frac{\lambda}{i+j+1} \right) \xi_j = \frac{\lambda}{i+1} u_0$$

for $i = 1, 2, \ldots, q$. We are given λ and u_0, and we can choose a value for q, the number of basis functions in our approximation. For the case $\lambda = 1, u_0 = 1, q = 3$ I used the computer algebra system *Maxima* to compute the coefficients, arriving at the solution

$$U(t) = \frac{35}{116}t^3 + \frac{45}{116}t^2 + \frac{30}{29}t + 1.$$

References

1. K. Ericsson, D. Estep, P. Hansbo, and C. Johnson, *Computational differential equations*, Cambridge University Press, 1996.

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