SUMMER EXAMINATIONS 2010

Exam Code(s)	3BS9, 4BS2
Exam(s)	Third and Fourth Science
Module	Computing Techniques of Applied Mathematics
Module Code	CS 305
External Examiner	Prof. D. G.C Mckeon
Internal Examiners	Dr. Michael Tuite, Dr. David O'Keeffe
Instructions	**Full marks for THREE completed questions**
Duration	2 Hours
No. of Pages	6 pages
Department	Applied Mathematics

Requirements:

Log Tables	Yes
Other Materials	Students may use their own electronic calculators which must not be capable of storing text.

1. Consider the following two point boundary value problem for a function $u = u(x)$, which is a solution of :

$$-\frac{d^2u}{dx^2} + 4u = x \quad \text{for} \quad 0 < x < 2$$

$$u = 0 \quad \text{for} \quad x = 0, 2$$

(a) Show that a weak form of the boundary value problem is given by

$$B(u, w) = \rho(w)$$

where

$$B(u, w) = \int_0^2 \left(\frac{du}{dx}\frac{dw}{dx} + 4uw \right) dx \quad \text{and} \quad \rho(w) = \int_0^2 xw dx$$

and where $w(x)$ is a sufficiently smooth weight function. In your derivation, you should classify the boundary conditions.

(b) Seeking an approximate solution of the form

$$U_N = \phi_0 + \sum_{j=1}^{N} c_j \phi_j$$

where $\{\phi_i\}_{i=1}^N$ are pre-selected approximation functions, show that for the above procedure the $\{c_i\}_{i=1}^N$ are determined by solving the linear equations

$$\sum_{j=1}^{N} B_{ij} c_j = F_i \quad \text{for} \quad i = 1...N$$

where

$$B_{ij} = B(\phi_i, \phi_j) \quad \text{and } F_i = \rho(\phi_i) - B(\phi_0, \phi_i).$$

(c) Calculate the approximate solution for the above problem with $\phi_0 = 0,\ \phi_1 = x(2-x),\ \phi_2 = x^2(2-x)$.

2. Consider the following boundary value problem in a rectangular region Ω in the (x, y) plane:

$$-\nabla^2 u = x \text{ in } \Omega,$$
$$u = 0 \text{ on } \Gamma$$

where Ω is the rectangular region bounded by Γ, where Γ is defined by the lines $x = 0, 1$ and $y = 0, 2$.

(a) Show that the weak form of the boundary value problem is given by

$$B(u,\, w) = \rho(w)$$

where

$$B(u, w) = \int_\Omega \nabla u.\nabla w dxdy \quad \text{and} \quad \rho(w) = \int_\Omega xwdxdy$$

and where $w = w(x, y)$ is a sufficiently smooth weight function. In your derivation, you should classify the boundary conditions.

(b) Seeking an approximate solution to this problem of the form

$$U_N = \phi_0 + \sum_{j=1}^{N} c_j \phi_j$$

where $\{\phi_i\}_{i=1}^N$ are pre-selected approximation functions,then the $\{c_i\}_{i=1}^N$ are determined by solving the set of linear equations

$$\sum_{j=1}^{N} B_{ij} c_j = F_i \quad \text{for } i = 1, ..N$$

where $B_{ij} = B(\phi_i, \phi_j)$, $F_i = \rho(\phi_i) - B(\phi_0, \phi_j)$.

Calculate the approximation for the above problem with $N = 2$ with

$$\phi_0 = 0,\ \phi_1 = xy(2 - y)(1 - x),\ \phi_2 = x^2y^2(2 - y)(1 - x)$$

[**Hint**: You may quote the following identity

$$-\int_\Omega w\nabla^2 udxdy = -\int_\Gamma w\,\frac{\partial u}{\partial n}ds + \int_\Omega \nabla u.\nabla wdxdy$$

where we have the usual notation, and where Γ is the boundary of Ω.]

3. Consider the boundary value problem

$$-\tfrac{d^2u}{dx^2} + 3u = x \quad \text{for} \quad 0 < x < 1$$
$$\text{with } u(0) = u(1) = 0$$

Assume that the weak form of the problem reduces to

$$B(u,\, w) = \rho(w)$$

where

$$B(u,\, w) \;=\; \int_0^1 \left(\frac{du}{dx}\frac{dw}{dx} + 3uw\right) dx \quad \text{and} \quad \rho(w) = \int_0^1 xwdx$$

and where $w(x)$ is a sufficiently smooth weight function.

(a) Consider the region $[0,1]$ as four linear elements $(\Omega_k, k = 1...4)$ of equal length with nodes $x_0 = 0,\, x_1 = 1/4,\, x_2 = 1/2,\, x_3 = 3/4,\, x_4 = 1$ and using linear functions $\phi_i(x)$ such that

$$\phi_i(x) = \begin{cases} (x - x_{i-1})/(x_i - x_{i-1}), & \text{for } x_{i-1} \le x \le x_i \\ (x_{i+1} - x)/(x_{i+1} - x_i), & \text{for } x_i \le x \le x_{i+1} \\ \quad 0 \quad \text{elsewhere} & \end{cases}$$

Let the approximate solution be of the form $\displaystyle\sum_{k=0}^{4} c_k\phi_k$.

Show that in order to satisfy the boundary conditions $c_0 = c_4 = 0$ and so the Galerkin formulation of the problem reduces to

$$\sum_{j=1}^{3} K_{ij}c_j = F_i,\ i = 1, 2, 3$$

where

$$K_{ij} = \int_0^1 \left\{\frac{d\phi_i}{dx}\frac{d\phi_j}{dx} + 3\phi_i\phi_j\right\} dx,\ F_i = \int_0^1 x\phi_i dx\,.$$

(b) By considering the functions, as defined above, and using the fact that

$$\int_0^1 g(x)\,dx = \sum_{k=1}^{4}\int_{\Omega_k} g(x)\,dx$$

for any function $g(x)$, show that the problem reduces to the set of equations

$$\begin{bmatrix} 68 & -31 & 0 \\ -31 & 68 & -31 \\ 0 & -31 & 68 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 6 \\ 12 \\ 18 \end{bmatrix}$$

4. Consider the eigenvalue problem in $0 < x < 2$

$$\frac{d^2y}{dx^2} - \lambda y = 0$$

with $y(0) = y(2) = 0$.

(a) Show that the weak form of the problem is given by

$$A(\, u, v\,) = \lambda B(\, u,\, v\,)$$

where

$$A(y,\, v) = -\int_0^2 \frac{dy}{dx}\frac{dv}{dx}\, dx, \quad B(y,\, v) = \int_0^2 v(x)\, y(x)\, dx$$

and $v(x)$ is a sufficiently smooth function with $v(0) = v(2) = 0$.

(b) Approximating $y(x)$ by the function

$$y_N = \sum_{i=1}^{N} c_i \phi_i(x)$$

show that the problem reduces to finding the roots of the equation

$$\det(\mathbf{A} - \lambda \mathbf{B}) = 0$$

where the terms in the matrices $\mathbf{A}$, $\mathbf{B}$ are given by

$$A_{ij} = A(\phi_i, \phi_j) \quad \text{and} \quad B_{ij} = B(\phi_i, \phi_j).$$

(c) Letting $N = 2$ and

$$\phi_1 = x(2-x), \quad \phi_2 = x^2(2-x)$$

estimate the first two eigenvalues of the problem above.

5. Consider the following system of linear equations:

$$\begin{bmatrix} 1 & -3 & 10 \\ 31 & 8 & 1 \\ 2 & 1 & 68 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 9 \end{bmatrix}$$

(a) Write a MAPLE command to define the coefficient matrix **A** (say) in the above system of equations.

(b) Write MAPLE commands to solve for the matrix **X** in the following matrix equation:

$$\mathbf{X}\mathbf{A}^{\mathbf{T}} = \mathbf{B}$$

where $\mathbf{A}^{\mathbf{T}}$ denotes the transpose of **A**. You may assume the matrix **B** has being defined earlier.

(c) Write MAPLE commands to calculate the eigenvalues, the eigenvectors and characteristic polynomial of **A**.

(d) Write MAPLE commands to solve the linear system of equations defined.

(e) Write MAPLE commands to solve each of the following problems:

i.

$$\frac{du}{dx} + 2u = -3e^{-x}, \quad u(0) = 1$$

ii.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x\cos x, \quad y(0) = 1, \frac{dy}{dx}(0) = 0$$