Ollscoil na hÉireann, Gaillimh

National University of Ireland, Galway

Semester Two Examinations, 2007/2008

Exam Codes(s)	3CS2, 4BS2
Exam(s)	Third and Fourth Science
Module Code(s)	CS305
Module(s)	Computing techniques of applied mathematics
Repeat Paper	
External Examiner(s)	Professor J. Burzlaff
Internal Examiner(s)	Professor T.N. Sherry
	Dr. P. O'Leary
	Dr. T. Waters

Instructions: Answer any THREE questions.

Duration 2 hours

Requirements

Handout	
MCQ	
Statistical Tables	Yes - log tables
Graph paper	
Log graph paper	
Other material	

no. of pages	6
Department(s)	Mathematical Physics

1. Consider the following two point boundary value problem for a function $u = u(x)$, which is a solution of:

$$-\frac{d^2u}{dx^2} + 2u = x \quad \text{for} \quad 0 < x < 1$$

$$u = 0 \quad \text{for} \quad x = 0, 1$$

   (a) Show that a weak form of the boundary value problem is given by

$$B(u, w) = \rho(w)$$

   where

$$B(u, w) = \int_0^1 \left(\frac{du}{dx}\frac{dw}{dx} + 2uw \right) dx \quad \text{and} \quad \rho(w) = \int_0^1 xw dx$$

   and where $w(x)$ is a sufficiently smooth weight function. In your derivation, you should classify the boundary conditions on w.

   (b) Seeking an approximate solution of the form

$$u_N = \phi_0 + \sum_{i=1}^{N} c_i \phi_i$$

   where ϕ_i are pre-selected approximation functions, show that for the above procedure the c_i are determined by solving the linear equations

$$\sum_{i=1}^{N} B_{ij} c_i = F_j \quad \text{for} \quad j = 1 \dots N$$

   where

$$B_{ij} = B(\phi_i, \phi_j) \quad \text{and} \quad F_j = \rho(\phi_j) - B(\phi_0, \phi_j).$$

   (c) Calculate the approximate solution for the above problem with $\phi_0 = 0,\ \phi_1 = x(1 - x),\ \phi_2 = x^2(1 - x)$.

2. Consider the following boundary value problem in a rectangular region Ω in the (x, y) plane:

$$-\nabla^2 u = x \text{ in } \Omega,$$

$$u = 0 \text{ on } \Gamma$$

where Ω is the rectangular region bounded by Γ, where Γ is defined by the lines $x = 0, 2$ and $y = 0, 3$.

(a) Show that the weak form of the boundary value problem is given by

$$B(u, w) = \rho(w)$$

where

$$B(u, w) = \int_\Omega \nabla u.\nabla w \, dxdy \text{ and } \rho(w) = \int_\Omega xwdxdy$$

and where $w = w(x, y)$ is a sufficiently smooth weight function. In your derivation, you should classify the boundary conditions on w.

(b) Seeking an approximate solution to this problem of the form

$$u_N = \phi_0 + \sum_{i=1}^{N} c_i \phi_i$$

where ϕ_i are pre-selected approximation functions, show that the c_i are determined by solving the set of linear equations

$$\sum_{i=1}^{N} B_{ij} c_i = F_j \quad \text{for } j = 1, ..N$$

where $B_{ij} = B(\phi_i, \phi_j), F_j = \rho(\phi_j) - B(\phi_0, \phi_j)$.

Calculate *one* component of B and *one* component of F for the above problem with $\phi_0 = 0,\ \phi_1 = xy(3 - y)(2 - x), \phi_2 = x^2 y(3 - y)(2 - x)$.

[**Hint:** You may quote the following identity

$$-\int_\Omega w\nabla^2 udxdy = -\int_\Gamma w\frac{\partial u}{\partial n}ds + \int_\Omega \nabla u.\nabla wdxdy$$

where we have used the usual notation, and where Γ is the boundary of Ω.]

3. Consider the boundary value problem

$$-\frac{d^2u}{dx^2} + 2u = x \quad \text{for} \quad 0 < x < 1 \qquad \text{with} \quad u(0) = u(1) = 0$$

Assume that the weak form of the problem reduces to

$$B(u,w) = \rho(w)$$

where

$$B(u,w) = \int_0^1 \left(\frac{du}{dx}\frac{dw}{dx} + 2uw \right) dx \quad \text{and} \quad \rho(w) = \int_0^1 xw dx$$

and where $w(x)$ is a sufficiently smooth weight function.

(a) Consider the region [0,1] as four linear elements ($\Omega_k, k = 1 \ldots 4$) of equal length with nodes $x_0 = 0, x_1 = 1/4, x_2 = 1/2, x_3 = 3/4, x_4 = 1$ and using linear functions $\phi_i(x)$ such that

$$\phi_i(x) = \begin{cases} (x - x_{i-1})/(x_i - x_{i-1}), \text{ for } x_{i-1} \le x \le x_i \\ (x_{i+1} - x)/(x_{i+1} - x_i), \text{ for } x_i \le x \le x_{i+1} \\ \qquad\qquad 0 \text{ elsewhere} \end{cases}$$

Let the approximate solution be of the form $\sum_{k=0}^{4} c_k \phi_k$.

Show that in order to satisfy the boundary conditions we must set $c_0 = c_4 = 0$, and so the Galerkin formulation of the problem reduces to

$$\sum_{j=1}^{3} K_{ij} c_j = F_i, \quad i = 1, 2, 3$$

where

$$K_{ij} = \int_0^1 \left\{ \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} + 2\phi_i\phi_j \right\} dx, \quad F_i = \int_0^1 x\phi_i dx \ .$$

(b) By considering the functions, as defined above, and using the fact that

$$\int_0^1 g(x)dx = \sum_{k=1}^{4} \int_{\Omega_k} g(x)dx$$

for any function $g(x)$, show that the problem reduces to the set of equations

$$\begin{bmatrix} 200 & -98 & 0 \\ -98 & 200 & -98 \\ 0 & -98 & 200 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \frac{3}{2} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

4. Consider the eigenvalue problem in $0 < x < 3$

$$-\frac{d^2u}{dx^2} + \lambda u = 0$$

with $u(0) = u(3) = 0$

(a) Show that the weak form of the problem is given by

$$A(u, v) = \lambda B(u, v)$$

where

$$A(u, v) = \int_0^3 \frac{du}{dx}\frac{dv}{dx}dx; \qquad B(u, v) = -\int_0^3 uvdx$$

and $v(x)$ is a sufficiently smooth function with $v(0) = v(3) = 0$.

(b) Approximating $u(x)$ by the function

$$u_N = \sum_{k=0}^{N} c_k\phi_k(x)$$

show that the problem reduces to finding the roots of the equation

$$\det(A - \lambda B) = 0$$

where the elements of the matrices A, B are given by

$$A_{ij} = A(\phi_i, \phi_j) \text{ and } B_{ij} = B(\phi_i, \phi_j).$$

(c) Letting $N = 2$ and

$$\phi_1 = x(3 - x), \phi_2 = x^2(3 - x)$$

estimate the first two eigenvalues of the problem above.

5. Consider the following system of linear equations:

$$\begin{bmatrix} 2 & -1 & 1 \\ 2 & 9 & 4 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(a) Write a MAPLE command to define the coefficient matrix A (say) in the above system of equations

(b) Write MAPLE commands to calculate the eigenvalues and eigenvectors of A.

(c) Write MAPLE commands to calculate the characteristic polynomial of A.

(d) Write MAPLE commands to solve the linear system of equations defined.

(e) Write MAPLE commands to solve each of the following problems:

i.

$$\frac{dy}{dt} + y = 9e^{-2t}, y(0) = 2,$$

ii.

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 5y = \cos t, y(0) = 1, \frac{dy}{dt}(0) = 0.$$