Example 1.3

Solve the first order linear differential equation

$$\frac{dy}{dx} - y = e^x$$

using the integrating factor method.

1.)

$$I.F. = e^{\int P(x)dx} = e^{\int -1dx} = e^{-x}$$

(note that we do not include the arbitrary constant C).

2.) Multiply the D.E. by the I.F. to get

$$e^{-x}(\frac{dy}{dx} - y) = e^x \cdot e^{-x} \quad (\star)$$

3.) Recall that the L.H.S. of $(\star)$ equals

$$\frac{d}{dx}(ye^{-x})$$

therefore

$$\frac{d}{dx}(ye^{-x}) = 1 \Rightarrow d(ye^{-x}) = 1dx$$

4.) Integrate and solve for y:

$$\int d(ye^{-x}) = \int 1dx \Rightarrow ye^{-x} = x + C \Rightarrow y = xe^{x} + c$$

Example 1.4

Solve the first order linear differential equation

$$\frac{dy}{dx} + 2xy = x$$

using the integrating factor method.

1. Note that $P(x) = 2x$ while $Q(x) = x$.
2. $$I.F. = e^{\int P(x)dx} = e^{\int 2xdx} = e^{x^2}$$

   (note that we do not include the arbitrary constant C).
3. Multiply the D.E. by the I.F. to get

$$e^{x^2}(\frac{dy}{dx} + 2xy) = e^{x^2}x = \frac{d}{dx}\left(ye^{x^2}\right)$$

$$\implies ye^{x^2} = \int e^{x^2} x dx = e^{x^2}/2 + C$$

$$\implies y = \frac{e^{x^2}/2 + C}{e^{x^2}} = \frac{e^{x^2} + C}{2e^{x^2}}$$

Example 1.5

Solve the first order linear differential equation

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{\sin x}{x^2}$$

$P(x) = 2/x$; $Q(x) = (\sin x)/x^2$, so we have

$$\int P(x)dx = \int \left(\frac{2}{x}\right) dx = 2 \int \left(\frac{1}{x}\right) dx = 2 \log x = \log(x^2)$$

Integrating Factor $= e^{\int P(x)dx} = e^{\log(x^2)} = x^2$

$$\implies x^2\left[\frac{dy}{dx} + \frac{2y}{x}\right] = x^2\left[\frac{\sin x}{x^2}\right]$$

$$\implies \frac{d}{dx}\left(yx^2\right) = \sin x$$

$$\implies yx^2 = \int \sin x dx = -\cos x + C$$

$$\implies y = \frac{-\cos x + C}{x^2}$$