

MA140-Engineering Calculus

Lecture 20

October 19, 2017

Example 1.1

Evaluate:

$$\int \frac{dx}{x(x^2 + 1)^2}$$

The form of the partial fraction decomposition is

$$\frac{1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}$$

Add up the fractions:

$$\begin{aligned} \frac{1}{x(x^2 + 1)^2} &= \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2} \\ &= \frac{A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x}{x(x^2 + 1)^2} \end{aligned}$$

So:

$$1 = A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x$$

$$\Rightarrow 1 = A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex$$

$$\Rightarrow 1 = (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A$$

If we equate coefficients, we get the system

$$\left\{ \begin{array}{rcl} A + B & = & 0 \\ C & = & 0 \\ 2A + B + D & = & 0 \\ C + E & = & 0 \\ A & = & 1 \end{array} \right.$$

Solving this system gives: $A = 1, B = -1, C = 0, D = -1, E = 0$

Thus

$$\begin{aligned}
 \int \frac{dx}{x(x^2 + 1)^2} &= \int \left(\frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2} \right) dx \\
 &= \int \left(\frac{1}{x} + \frac{-x + 0}{x^2 + 1} + \frac{-x + 0}{(x^2 + 1)^2} \right) dx \\
 &= \int \left(\frac{1}{x} + \frac{-x}{x^2 + 1} + \frac{-x}{(x^2 + 1)^2} \right) dx \\
 &= \underbrace{\int \frac{1}{x} dx}_{I_1} + \underbrace{\int \frac{-x}{x^2 + 1} dx}_{I_2} + \underbrace{\int \frac{-x}{(x^2 + 1)^2} dx}_{I_3}
 \end{aligned}$$

$$I_1 = \int \frac{1}{x} dx$$

Therefore

$$I_1 = \ln(x) + c_1$$

Also

$$I_2 = \int \frac{-x}{x^2 + 1} dx$$

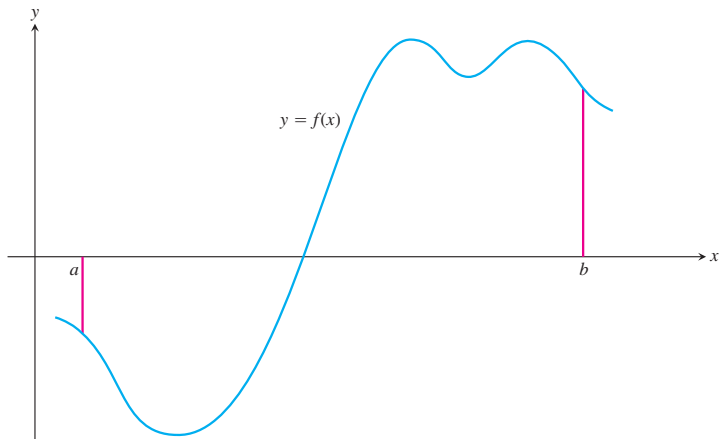
Let $u = x^2 + 1$ then $du = 2x dx$, so $-\frac{du}{2} = -x dx$

$$\int \frac{-x}{x^2 + 1} dx = \int \frac{-du}{2u} = \frac{-1}{2} \int \frac{du}{u} = \frac{-1}{2} \ln u + c_2 = \frac{-1}{2} \ln(x^2 + 1) + c_2$$

$$I_3 = \int \frac{-x}{(x^2 + 1)^2} dx$$

Let $u = x^2 + 1$, then $du = 2x dx$, so $\frac{-du}{2} = -x dx$, Therefore:

$$\begin{aligned} I_3 &= \int \frac{-x}{(x^2 + 1)^2} dx = \int \frac{-du}{2(u)^2} = \frac{-1}{2} \int \frac{du}{u^2} = \frac{-1}{2} \int u^{-2} du \\ &= \frac{-1}{2} \left(\frac{u^{-2+1}}{-2+1} \right) + c_3 = \frac{-1}{2} \left(\frac{(x^2 + 1)^{-2+1}}{-2+1} \right) + c_3 = \frac{-1}{2} \left(\frac{(x^2 + 1)^{-1}}{-1} \right) + c_3 \end{aligned}$$



To do so, we choose $n - 1$ points $\{x_1, x_2, \dots, x_{n-1}\}$ between a and b and satisfying

$$a < x_1 < x_2 < \dots < x_{n-1} < b$$

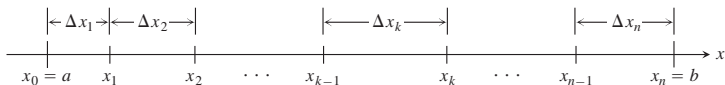
To make the notation consistent, we denote a by x_0 and b by x_n , so that :

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

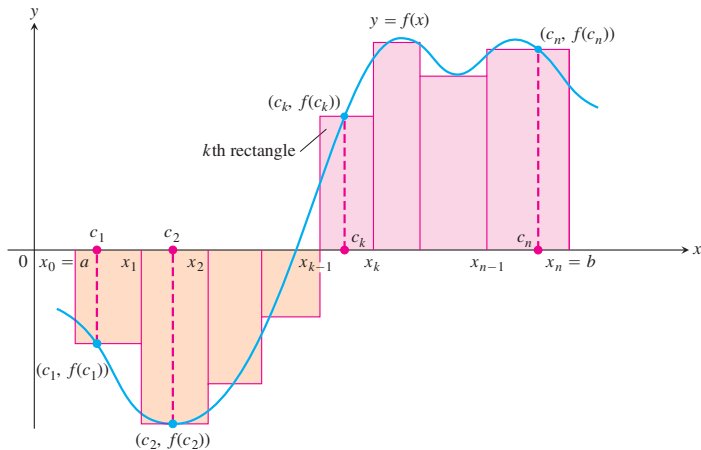
The set

$$P = \{x_0, x_1, x_2, \dots, x_{n-1}, x_n\}$$

is called a partition of $[a, b]$

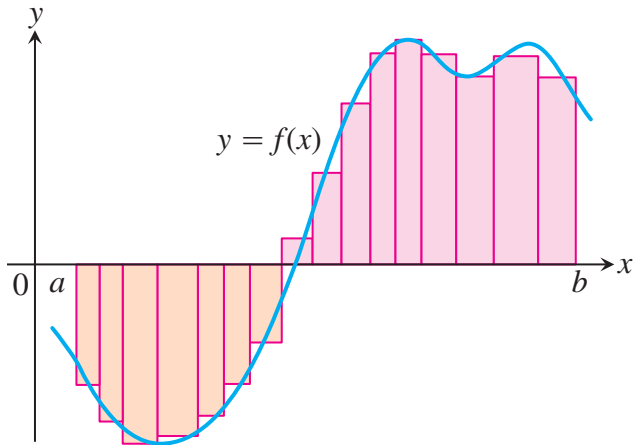


In each subinterval we select some point c_k . Then on each subinterval we stand a vertical rectangle that stretches from the x-axis to touch the curve at $(c_k, f(c_k))$:



We can also let all the subintervals to have equal widths, $\Delta x_k = \frac{1}{n}$
Then the area of each rectangle is

$$\Delta x_k \cdot f(c_k) = \frac{1}{n} \cdot f(c_k)$$

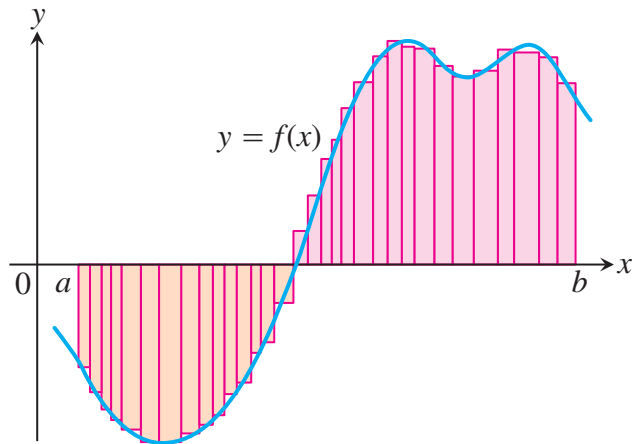


Finally we sum all these products to get

$$S_P = \sum_{k=1}^n \frac{1}{n} \cdot f(c_k)$$

S_P is called a Riemann sum for f on the interval $[a, b]$.

We can also make the width of the subintervals smaller.



If we make n larger (or make $\frac{1}{n}$ smaller), then

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} \cdot f(c_k)$$

will give us an accurate answer.

We say that the definite integral of f from a to b is:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} \cdot f(c_k)$$

The diagram illustrates the components of a definite integral $\int_a^b f(x) dx$. The integral sign is on the left, with a vertical line extending from it. The upper limit b is at the top of this line, and the lower limit a is at the bottom. The function $f(x)$ is written in the middle, and dx is at the end. A horizontal brace underneath the entire expression $\int_a^b f(x) dx$ is labeled "Integral of f from a to b ".

Upper limit of integration

Integral sign

Lower limit of integration

The function is the integrand.

x is the variable of integration.

When you find the value of the integral, you have evaluated the integral.

Integral of f from a to b

Theorem 1.2

The Fundamental Theorem of Calculus (1):

Suppose that f is a continuous function on $[a, b]$, and F is an anti-derivative of f . Then:

$$\int_a^b f(x)dx = F(x) \Big|_a^b = F(b) - F(a)$$

Example 1.3

Evaluate

$$\int_0^1 x^3 dx$$

An anti-derivative can be obtained by evaluating the following integral:

$$\int x^3 dx = x^4/4$$

So $F(x) = x^4/4$, therefore:

$$\int_0^1 x^3 dx = x^4/4 \Big|_0^1 = (1^4/4) - (0^4/4) = 1/4$$

Theorem 1.4

- *Order of integration:*

$$\int_a^b f(x)dx = - \int_b^a f(x)dx$$

- *Zero width interval*

$$\int_a^a f(x)dx = 0$$

- *if $a < c < b$, then*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Example 1.5

Suppose that $\int_{-1}^1 f(x)dx = 5$, $\int_1^4 f(x)dx = -2$ and $\int_{-1}^1 h(x)dx = 7$.
Then find



$$\int_4^1 f(x)dx$$



$$\int_{-1}^1 [2f(x) + 3h(x)]dx$$



$$\int_{-1}^4 f(x)dx$$

Theorem 1.6

The Fundamental Theorem of Calculus (2):

Suppose that f is a continuous function on $[a, b]$,

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

Example 1.7

Find

$$\frac{d}{dx} \int_0^x \sqrt{1+2t} dt$$

Recall that

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

so:

$$\frac{d}{dx} \int_0^x \sqrt{1+2t} dt = \sqrt{1+2x}$$

Example 1.8

Find $\frac{dy}{dx}$, if

$$y = \int_a^x \cos(t) dt$$

By The fundamental theorem of calculus, we know that:

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

So:

$$\frac{d}{dx} \int_a^x \cos(t) dt = \cos(x)$$