MA140-Engineering Calculus

Lecture 17

October 15, 2017

Solving Applied Optimization Problems:

(1) *Read the problem.* Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?

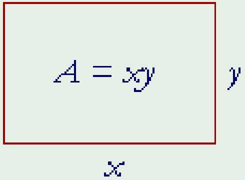
(2) *Draw a picture.* Label any part that may be important to the problem.

(3) *Introduce variables.* List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.

(4) *Write an equation for the unknown quantity.* If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.

Example 1.1

A rectangle has the following side lengths



Find x and y if the area is to be maximized if the perimeter equals 30m.

The perimeter equals $30m$, it means $2(x+y)=30$ or $x+y=15$, so

$$y = 15 - x, (\star)$$

$Area = A = x \cdot y$, Using $(\star)$, we have

$$A = x(15-x)$$

Now A is a function of x, in order to find the maximum value of A, we differentiate it:

$$\frac{dA}{dx} = 15 - 2x$$

Let:

$$\frac{dA}{dx} = 0 \Rightarrow x = \frac{15}{2} (y = \frac{15}{2})$$

So $x = \frac{15}{2}$ is a critical point.

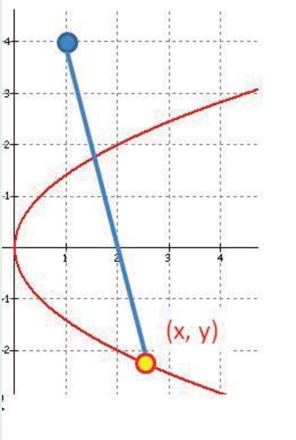
Now we use the second derivative test

$$\frac{d^2A}{dx^2} = 2 < 0$$

so $x = \frac{15}{2}$ is a local maximum

Example 1.2

Find the point on the parabola $y^2 = 2x$, closest to the point $(1, 4)$



The distance between the two points (x, y) and $(1, 4)$ equals:

$$d = \sqrt{(x-1)^2 + (y-4)^2}$$

Instead of minimizing d we can minimize distance squared.

$$r = (x-1)^2 + (y-4)^2$$

As the coordinates of the point satisfy $y^2 = 2x$ or $\frac{y^2}{2} = x$, let $y = t$ then $x = \frac{t^2}{2}$. So $(x, y) = (\frac{t^2}{2}, t)$ and $(1, 4)$ have distance squared equal to

$$r(t) = (\frac{t^2}{2} - 1)^2 + (t-4)^2$$

$$r(t) = \frac{t^4}{4} - t^2 + 1 + t^2 + 16 - 8t = \frac{t^4}{4} - 8t + 17$$

$$r'(t) = t^3 - 8 = 0 \Rightarrow r'(t) = (t-2)(t^2 + 2t + 4) = 0$$

So $t = 2$ is the only critical point as the other factor does not have any real roots

$$d''(t) = 3t^2 \Rightarrow d''(2) = 12 > 0$$

So $t = 2$ is a local minimum, therefore $(2, 2)$ is the point on $y^2 = 2x$ closest to $(1, 4)$

suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and $g'(a) \neq 0$. Then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Proof: Working backward from $f'(a)$ and $g'(a)$, which are themselves limits, we have

$$\frac{f'(a)}{g'(a)} = \frac{\lim_{x \to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x \to a} \frac{g(x)-g(a)}{x-a}} = \lim_{x \to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{f(x) - 0}{g(x) - 0} = \lim_{x \to a} \frac{f(x)}{g(x)}$$

Theorem 1.3

L'Hôpital's Rule:

If f *and* g *are differentiable and* $g'(x) \neq 0$ *near* a *(except possibly at* a*),*

- *if* $\lim_{x \to a} f(x) = 0$ *and* $\lim_{x \to a} g(x) = 0$
  or
- $\lim_{x \to a} f(x) = \pm\infty$ *and* $\lim_{x \to a} g(x) = \pm\infty$

Then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Example 1.4

Find the following limit:

$$\lim_{x \to 1} \frac{\ln x}{x - 1}$$

As $\lim_{x \to 1}(\ln x) = \ln(1) = 0$ and $\lim_{x \to 1}(x - 1) = 0$, we can apply L'Hôpital's Rule:

$$\lim_{x \to 1} \frac{\ln x}{x - 1} \overset{\text{H}}{=} \lim_{x \to 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x - 1)} = \lim_{x \to 1} \frac{\frac{1}{x}}{1} = \lim_{x \to 1} \frac{1}{x} = \frac{1}{1} = 1$$

Example 1.5

Find

$$\lim_{x\to\infty} \frac{e^x}{x^2}$$

$\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty} x^2 = \infty$, so we can apply the L'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{e^x}{x^2} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{2x}$$

This result is also of the form $\frac{\infty}{\infty}$, so we can apply the L'Hôpital's Rule again:

$$\lim_{x\to\infty} \frac{e^x}{2x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

Example 1.6

Find

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3}$$

$\lim_{x\to 0}(\sin(x) - x) = 0$ and $\lim_{x\to 0} x^3 = 0$, as we see the limit is of the form $\frac{0}{0}$. so

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3} \overset{\text{H}}{=} \lim_{x\to 0} \frac{\cos(x) - 1}{3x^2}$$

Note that $\lim_{x\to 0}(\cos x - 1) = 0$ and $\lim_{x\to 0}(3x^2) = 0$, so again we get $\frac{0}{0}$, so we can to apply L'Hôpital's Rule again,

$$\lim_{x\to 0} \frac{\cos(x) - 1}{3x^2} \overset{\text{H}}{=} \lim_{x\to 0} \frac{-\sin x}{6x} = \frac{0}{0}$$

We can use L'Hôpital's Rule again:

$$\lim_{x\to 0} \frac{-\sin x}{6x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{-\cos x}{6} = \frac{-1}{6}$$