

Tangents

Example 1.1

Determine the (equation of the) tangent (line) to the function

$$f(x) = x^2$$

at the point $x = 5$.

$$f(x) = x^2 \implies f'(x) = 2x$$

$$\text{At the point } x = 5, y = f(x) = f(5) = 5^2 = 25. \quad (1)$$

$$\text{At the point } x = 5, f'(x) = f'(5) = (2)(5) = 10.$$

This means, the tangent line has slope 10 and goes through the point $(5, 25)$. The equation of a line with slope m is $y = mx + c$. So here we have $y = 10x + c$. Since the line must go through $(5, 25)$, we insert this point in to the equation to get $25 = (10)(5) + c \implies c = -25$. Finally our answer is $y = 10x - 25$.

Example 1.2

Find the points on the curve $y = 4x^3 - 6x$ where the tangent (line) is parallel to $2x + y + 5 = 0$.

To say two lines are “parallel” means they have the same slope:

- 1 What is the slope of the line? Rewrite it as $y = -2x - 5$, so the slope is -2 . (The slope of a line $y = mx + c$ is m .)
- 2 What is the slope at any point on the curve?
 $y = 4x^3 - 6x \implies y' = 12x^2 - 6$.

So we just set these two slopes equal to one another:

$$-2 = 12x^2 - 6 \implies 4 = 12x^2 \implies x = 1/\sqrt{3} \text{ or } x = -1/\sqrt{3}.$$

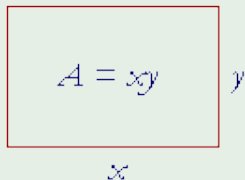
- $x = 1/\sqrt{3} \implies y = 4(1/\sqrt{3})^3 - 6/\sqrt{3} = -14/(3\sqrt{3})$
- $x = -1/\sqrt{3} \implies y = -4(1/\sqrt{3})^3 + 6/\sqrt{3} = 14/(3\sqrt{3})$

Solving Applied Optimization Problems:

- (1) *Read the problem.* Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?
- (2) *Draw a picture.* Label any part that may be important to the problem.
- (3) *Introduce variables.* List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.
- (4) *Write an equation for the unknown quantity.* If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.

Example 1.1

A rectangle has the following side lengths



Find x and y if the area is to be maximized if the perimeter equals 30m.

The perimeter equals $30m$, it means $2(x + y) = 30$ or $x + y = 15$, so

$$y = 15 - x, (\star)$$

Area = $A = x \cdot y$, Using (\star) , we have

$$A = x(15 - x)$$

Now A is a function of x , in order to find the maximum value of A , we differentiate it:

$$\frac{dA}{dx} = 15 - 2x$$

Let:

$$\frac{dA}{dx} = 0 \Rightarrow x = \frac{15}{2} (y = \frac{15}{2})$$

So $x = \frac{15}{2}$ is a critical point.

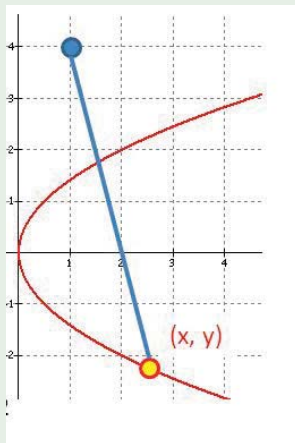
Now we use the second derivative test

$$\frac{d^2 A}{dx^2} = 2 < 0$$

so $x = \frac{15}{2}$ is a local maximum

Example 1.2

Find the point on the parabola $y^2 = 2x$, closest to the point $(1, 4)$



The distance between the two points (x, y) and $(1, 4)$ equals:

$$d = \sqrt{(x - 1)^2 + (y - 4)^2}$$

Instead of minimizing d we can minimize distance squared.

$$r = (x - 1)^2 + (y - 4)^2$$

As the coordinates of the point satisfy $y^2 = 2x$ or $\frac{y^2}{2} = x$, let $y = t$ then $x = \frac{t^2}{2}$. So $(x, y) = (\frac{t^2}{2}, t)$ and $(1, 4)$ have distance squared equal to

$$r(t) = \left(\frac{t^2}{2} - 1\right)^2 + (t - 4)^2$$

$$r(t) = \frac{t^4}{4} - t^2 + 1 + t^2 + 16 - 8t = \frac{t^4}{4} - 8t + 17$$

$$r'(t) = t^3 - 8 = 0$$

So $t = 2$ is the only critical point.

$$r''(t) = 3t^2 \Rightarrow r''(2) = 12 > 0$$

So $t = 2$ is a local minimum, therefore $(2, 2)$ is the point on $y^2 = 2x$ closest to $(1, 4)$

Example 1.3

Suppose a mug has the shape of a cylinder, and is being filled with water. At a particular point in time, the height of the water is rising at a rate of 1cm per second. If the circumference of the mug is 6π cm, what is the rate of increase of (the volume of) water in the mug?

$$V = \pi r^2 h. \text{ Circumference} = 2\pi r = 6\pi \implies r = 3 \implies V = 9\pi h.$$

$$\frac{dV}{dt} = 9\pi \frac{dh}{dt} = 9\pi(1) = 9\pi \text{ (cubic centimetres per second)}$$

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and $g'(a) \neq 0$. Then:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Proof: Working backward from $f'(a)$ and $g'(a)$, which are themselves limits, we have

$$\begin{aligned} \frac{f'(a)}{g'(a)} &= \frac{\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}}{\lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a}} = \lim_{x \rightarrow a} \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}} \\ &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \rightarrow a} \frac{f(x) - 0}{g(x) - 0} = \lim_{x \rightarrow a} \frac{f(x)}{g(x)} \end{aligned}$$

Theorem 1.3

L'Hôpital's Rule:

If f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a),

- if $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$
or
- $\lim_{x \rightarrow a} f(x) = \pm\infty$ and $\lim_{x \rightarrow a} g(x) = \pm\infty$

Then:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Example 1.4

Find the following limit:

$$\lim_{x \rightarrow 1} \frac{\ln x}{x - 1}$$

As $\lim_{x \rightarrow 1} (\ln x) = \ln(1) = 0$ and $\lim_{x \rightarrow 1} (x - 1) = 0$, we can apply L'Hôpital's Rule:

$$\lim_{x \rightarrow 1} \frac{\ln x}{x - 1} \stackrel{\text{H}}{=} \lim_{x \rightarrow 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x - 1)} = \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{1} = \lim_{x \rightarrow 1} \frac{1}{x} = \frac{1}{1} = 1$$

Example 1.5

Find

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$$

$\lim_{x \rightarrow \infty} e^x = \infty$ and $\lim_{x \rightarrow \infty} x^2 = \infty$, so we can apply the L'Hôpital's Rule:

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} \stackrel{\text{H}}{=} \lim_{x \rightarrow \infty} \frac{e^x}{2x}$$

This result is also of the form $\frac{\infty}{\infty}$, so we can apply the L'Hôpital's Rule again:

$$\lim_{x \rightarrow \infty} \frac{e^x}{2x} \stackrel{\text{H}}{=} \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty$$

Example 1.6

Find

$$\lim_{x \rightarrow 0} \frac{\sin(x) - x}{x^3}$$

$\lim_{x \rightarrow 0} (\sin(x) - x) = 0$ and $\lim_{x \rightarrow 0} x^3 = 0$, as we see the limit is of the form $\frac{0}{0}$. so

$$\lim_{x \rightarrow 0} \frac{\sin(x) - x}{x^3} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\cos(x) - 1}{3x^2}$$

Note that $\lim_{x \rightarrow 0} (\cos x - 1) = 0$ and $\lim_{x \rightarrow 0} (3x^2) = 0$, so again we get $\frac{0}{0}$, so we can to apply L'Hôpital's Rule again,

$$\lim_{x \rightarrow 0} \frac{\cos(x) - 1}{3x^2} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{-\sin x}{6x} = \frac{0}{0}$$

We can use L'Hôpital's Rule again:

$$\lim_{x \rightarrow 0} \frac{-\sin x}{6x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{-\cos x}{6} = \frac{-1}{6}$$