

# MA140-Engineering Calculus

## Lecture 31

November 14, 2017

Centers of Mass:  
Masses Along a Line:



Each mass  $m_k$  exerts a downward force  $m_k g$ .

Each of these forces has a tendency to turn the axis about the origin. This turning effect is called a torque.

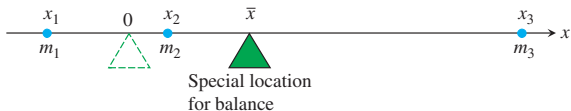
The torque of each mass is:

$$m_k g x_k$$

$$\begin{aligned} \text{System torque} &= m_1 g x_1 + m_2 g x_2 + m_3 g x_3 \\ &= g \cdot (m_1 x_1 + m_2 x_2 + m_3 x_3) \end{aligned}$$

The number  $m_1 x_1 + m_2 x_2 + m_3 x_3$  is called the moment of the system about the origin.

We usually want to know where to place the fulcrum to make the system balance, that is, at what point  $\bar{x}$  to place it to make the torques add to zero.



The torque of each mass about the fulcrum in this special location is:

$$(x_k - \bar{x})m_k g$$

So the torque of the new system is

$$\sum_{k=1}^3 (x_k - \bar{x})m_k g$$

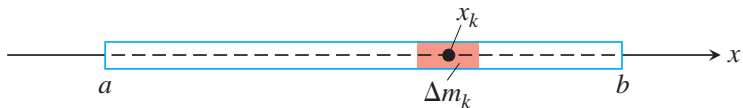
Then:

$$\begin{aligned} \sum_{k=1}^3 (x_k - \bar{x})m_k g = 0 &\Rightarrow g \sum_{k=1}^3 (x_k - \bar{x})m_k = 0 \\ \Rightarrow \sum_{k=1}^3 (m_k x_k - m_k \bar{x}) = 0 &\Rightarrow \sum_{k=1}^3 m_k x_k - \sum_{k=1}^3 m_k \bar{x} = 0 \\ \Rightarrow \sum_{k=1}^3 m_k x_k = \bar{x} \sum_{k=1}^3 m_k &\Rightarrow \bar{x} = \frac{\sum_{k=1}^3 m_k x_k}{\sum_{k=1}^3 m_k} \end{aligned}$$

This last equation tells us to find  $\bar{x}$  by dividing the system's moment about the origin by the system's total mass:

$$\Rightarrow \bar{x} = \frac{\sum_{k=1}^3 m_k x_k}{\sum_{k=1}^3 m_k} = \frac{\text{system moment about origin}}{\text{system mass}}$$

## Wires and Thin strips:



$$\bar{x} \cong \frac{\text{system moment}}{\text{system mass}}$$

The system mass is

$$\sum_{k=1}^n \Delta m_k$$

the moment of each piece  $x_k \Delta m_k$ , so

$$\text{System moment} \cong \sum_{k=1}^n x_k \Delta m_k$$

if the density of the strip at  $x_k$  is  $\delta(x_k)$ , expressed in terms of mass per unit length and if  $\delta$  is continuous, then  $\Delta m_k$  is approximately equal to  $\delta(x_k) \Delta x_k$ :

$$\Delta m_k \cong \delta(x_k) \Delta x$$

Therefore:

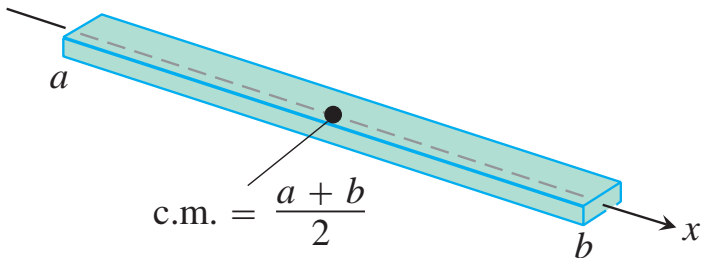
$$\bar{x} \cong \frac{\text{system moment}}{\text{system mass}} \cong \frac{\sum_{k=1}^n x_k \Delta m_k}{\sum_{k=1}^n \Delta m_k} \cong \frac{\sum_{k=1}^n x_k \delta(x_k) \Delta x}{\sum_{k=1}^n \delta(x_k) \Delta x}$$

The sums is a Riemann sum so when  $\Delta x \rightarrow 0$  or  $n \rightarrow \infty$ , then

$$\bar{x} = \frac{\int_a^b x \delta(x) dx}{\int_a^b \delta(x) dx}$$

### Example 1.1

Show that the center of mass of a straight, thin strip or rod of constant density lies halfway between its two ends.



We model the strip as a portion of the  $x$ -axis from  $x = a$  to  $x = b$ . We know that

$$\bar{x} = \frac{\int_a^b x\delta(x)dx}{\int_a^b \delta(x)dx}$$

the density is constant so  $\delta(x) = \delta$ . The numerator is:

$$\int_a^b x\delta dx = \delta \int_a^b x dx = \delta \left[ \frac{1}{2}x^2 \right]_a^b = \frac{\delta}{2}(b^2 - a^2)$$

The denominator is:

$$\int_a^b \delta dx = \delta \int_a^b dx = \delta x \Big|_a^b = \delta(b - a)$$

So:

$$\bar{x} = \frac{\int_a^b x\delta(x)dx}{\int_a^b \delta(x)dx} = \frac{a + b}{2}$$

When the density is constant, we call the center of mass the centroid of the object. To find the centroid we set  $\delta = 1$