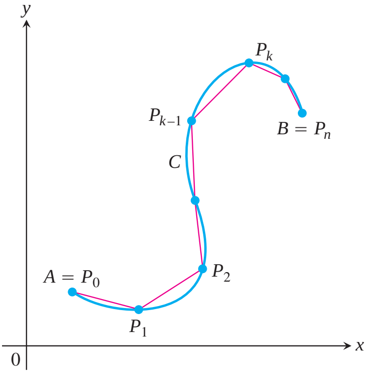
MA140-Engineering Calculus

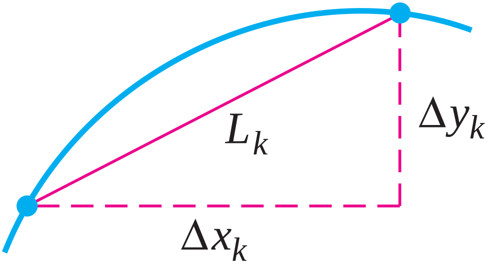
Lecture 28

November 7, 2017

Arc Length: Let C be a curve given by the equation $y = f(x)$. It may be helpful to imagine the curve as the path of a particle moving from point A to pint B. We subdivide the path (or arc) AB into n pieces at points $A = P_0, P_1, P_2, \cdots, P_n = B$. Join successive points of this subdivision by straight line segments



A representative line segment:



has length:

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

An intuitive approximation to the length of the curve AB, S, is the sum of all the lengths L_k :

$$S \cong \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

So

$$S \cong \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 (1 + (\frac{\Delta y_k^2}{\Delta x_k^2}))}$$

So:

$$S \cong \sum_{k=1}^{n} \sqrt{(1 + (\frac{\Delta y_k^2}{\Delta x_k^2}))}(\Delta x_k)$$

Using a Riemann sum approach.
Let $\Delta x \to 0$ or $n \to \infty$, we get:

$$S = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{(1 + (\frac{\Delta y_k^2}{\Delta x_k^2}))}(\Delta x_k)$$

So:

$$S = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx$$

Theorem 1.1

If f is continuously differentiable on the closed interval $[a, b]$, the length of the curve (graph) $y = f(x)$ from $x = a$ to $x = b$ is:

$$S = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

Example 1.2

Find the length of the curve

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \quad 0 \leq x \leq 1$$

We use the theorem with $a = 0$, $b = 1$, and

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1$$

First we take the derivative of y:

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}x^{1/2} = 2\sqrt{2}x^{1/2}$$

So

$$(\frac{dy}{dx})^2 = (2\sqrt{2}x^{1/2})^2 = 8x$$

The length of the curve from $x = 0$ to $x = 1$ is:

$$S = \int_0^1 \sqrt{1 + (\frac{dy}{dx})^2} dx = \int_0^1 \sqrt{1 + 8x} dx$$

$$= \frac{2}{3} \cdot \frac{1}{8} (1 + 8x)^{3/2} \Big]_0^1 = \frac{13}{6}$$

Example 1.3

The cable of a suspension bridge takes the shape of the curve:

$$y = \frac{h}{l^2}x^2 - \frac{2h}{l}x + h$$

Where $0 \leq x \leq 2l$, $h > 0$. Find the length of the cable.

$$\frac{dy}{dx} = \frac{2h}{l^2}x - \frac{2h}{l} = \frac{2h}{l}(\frac{x}{l} - 1)$$

So:

$$(\frac{dy}{dx})^2 = \left[\frac{2h}{l}(\frac{x}{l} - 1)\right]^2$$

The length of the curve equals:

$$S = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}dx = \int_0^{2l} \sqrt{1 + \left[\frac{2h}{l}(\frac{x}{l} - 1)\right]^2}dx$$

First we find the following indefinite integral:

$$\int \sqrt{1 + \left[\frac{2h}{l}(\frac{x}{l} - 1)\right]^2}dx, \quad (\star)$$

Let $u = \frac{2h}{l}(\frac{x}{l} - 1)$, then $du = \frac{2h}{l^2}dx$ or $\frac{l^2}{2h}du = dx$
So:

$$(\star) = \frac{l^2}{2h}\int \sqrt{1+u^2}du$$

quick reminder:

- $\sinh(x) = \frac{e^x - e^{-x}}{2}$
- $\cosh(x) = \frac{e^x + e^{-x}}{2}$
- $\frac{d}{dx}\sinh(x) = \cosh(x)$
- $\frac{d}{dx}\cosh(x) = \sinh(x)$
- $\cosh^2(x) - \sinh^2(x) = 1$