MA140-Engineering Calculus

Lecture 19

October 19, 2017

Definition 1.1

Rational functions have the general form

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials.

- **IF** degree of $p(x) <$ degree of $q(x)$, then $f(x)$ is a strictly proper rational function.
- **IF** degree of $p(x) =$ degree of $q(x)$, then $f(x)$ is a proper rational function.
- **IF** degree of $p(x) >$ degree of $q(x)$, then $f(x)$ is an improper rational function.

An improper or proper rational function can be expressed in terms of a strictly proper rational function

Example 1.2

Express $f(x) = \dfrac{3x^4 + 2x^3 - 5x^2 + 6x - 7}{x^2 - 2x + 3}$ in terms of a strictly proper rational function

$$f(x) = 3x^2 + 8x + 2 - \frac{14x + 13}{x^2 - 2x + 3}$$

(1) Linear factor to the power of 1:

A linear factor $(x-a)$ gives rise to the partial fraction of the form

$$\frac{A}{x-a}$$

(2) Linear factor to the power of greater than 1:

If $(x-\alpha)^k$ appears in the denominator, it will give rise to the following terms:

$$\frac{A_1}{x-\alpha}+\frac{A_2}{(x-\alpha)^2}+\cdots+\frac{A_k}{(x-\alpha)^k}$$

(3) Irreducible quadratic factors:
An irreducible quadratic $ax^2 + bx + c$ gives rise to partial fractions of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

(4) Irreducible quadratic factors to the power of greater than 1:
If $(ax^2 + bx + c)^k$ appears in the denominator, it will give rise to the following terms:

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$$

Example 1.3

Evaluate

$$\int \frac{3x+4}{x^2+7x+12}dx$$

$x^2+7x+12=(x+4)(x+3)$, so the partial fraction decomposition has the form:

$$\frac{3x+4}{x^2+7x+12}=\frac{A}{x+4}+\frac{B}{x+3}$$

To find the values of the undetermined coefficients A and B, we add the new fractions:

$$\frac{3x+4}{x^2+7x+12}=\frac{A}{x+4}+\frac{B}{x+3}=\frac{A(x+3)+B(x+4)}{(x+4)(x+3)}$$

The denominators on both sides of the above equation are identical, so the numerators must be equal:

$$3x+4 = A(x+3)+B(x+4) = Ax+3A+Bx+4B = (A+B)x+(3A+4B)$$

which means:

$$\begin{cases} A+B &= 3 \\ 3A+4B &= 4 \end{cases}$$

So $A = 3 - B$ so inserting this into the second equation, we get:

$$3(3-B)+4B = 4 \Rightarrow 9-3B+4B = 4 \Rightarrow 9+B = 4 \Rightarrow B = 4-9 \Rightarrow B = -5$$

Therefore $A = 3 - B = 3 + 5 = 8$.
So

$$\frac{3x+4}{x^2+7x+12} = \frac{8}{x+4} + \frac{-5}{x+3}$$

Now we can express the integral as:

$$\int \frac{3x+4}{x^2+7x+12}dx = \int \Big(\frac{8}{x+4} + \frac{-5}{x+3}\Big)dx$$

Using the sum rule, we have:

$$\int \frac{3x+4}{x^2+7x+12}dx = \int \Big(\frac{8}{x+4} + \frac{-5}{x+3}\Big)dx = \underbrace{\int \frac{8}{x+4}dx}_{I_1} + \underbrace{\int \frac{-5}{x+3}dx}_{I_2}$$

First we find I_1:

$$I_1 = \int \frac{8}{x+4} dx = 8 \int \frac{1}{x+4} dx$$

Let $u = x + 4$, then $du = dx$, so:

$$I_1 = \int \frac{8}{x+4} dx = 8 \int \frac{1}{x+4} dx = 8 \int \frac{du}{u} = 8 \ln(u) + c_1 = 8 \ln(x+4) + c_1$$

Therefore $I_1 = 8 \ln(x+4) + c_1$

$$I_2 = \int \frac{-5}{x+3} dx$$

Let $u = x + 3$, then $du = dx$, so:

$$I_2 = -5 \int \frac{1}{x+3} dx = -5 \int \frac{du}{u} = -5\ln(u) + c_2 = -5\ln(x+3) + c_2$$

Therefore $I_2 = -5\ln(x+3) + c_2$
So finally

$$\int \frac{3x+4}{x^2+7x+12} dx = \int \left(\frac{8}{x+4} + \frac{-5}{x+3}\right) dx = \underbrace{\int \frac{8}{x+4} dx}_{I_1} + \underbrace{\int \frac{-5}{x+3} dx}_{I_2}$$

$$= 8\ln(x+4) + c_1 - 5\ln(x+3) + c_2 = 8\ln(x+4) - 5\ln(x+3) + c$$