MA140-Engineering Calculus

Lecture 12

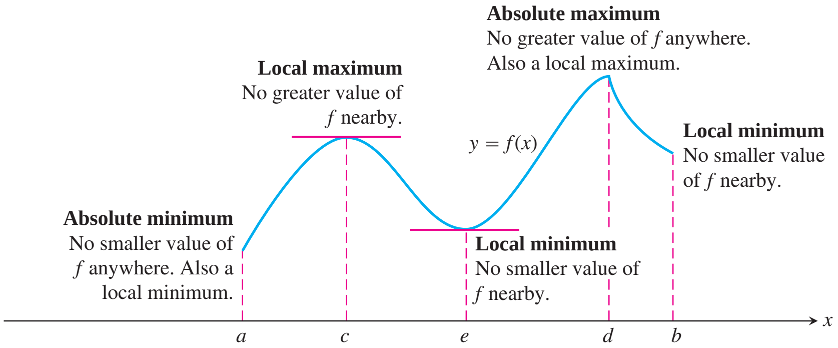
October 9, 2019

The second derivative

The derivative of the derivative is called the second derivative.
For a function $y = f(x)$, we write the second derivative as

$$\frac{d^2y}{dx^2} \quad or \quad f''(x) \quad or \quad f^{(2)}(x)$$

The basic idea is that the optimal value of a differentiable function $f(x)$ (its maximum and minimum value) generally occurs when $f'(x) = 0$


Absolute maximum
No greater value of f anywhere.
Also a local maximum.
Local maximum
No greater value of
f nearby.
y = f(x)
Local minimum
No smaller value
of f nearby.
Absolute minimum
No smaller value of
f anywhere. Also a
local minimum.
Local minimum
No smaller value of
f nearby.
x
a
c
e
d
b

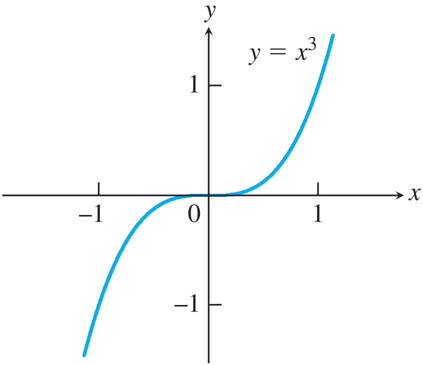
Definition 1.1

An interior point of the domain of a function f where f' is zero or undefined is a *critical point* of f.

Note: Some values of x satisfying $f'(x) = 0$ are not maximum or minimum.
for example $x = 0$ is a critical point of $f(x) = x^3$, because

$$f'(x) = 3x^2 = 0 \Rightarrow x = 0$$

But $x = 0$ is neither a maximum or minimum.



The First Derivative Test

We will show how to test the critical points of a function for the presence of local maximums and minimums.

Definition 1.2

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

- If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be *increasing* on I.
- If $f(x_1) > f(x_2)$ whenever $x_1 < x_2$, then f is said to be *decreasing* on I.

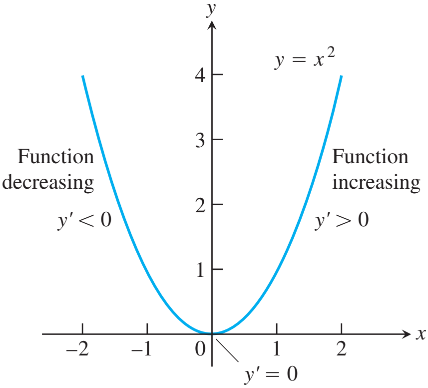
Example 1.3

The function $f(x) = x^2$ decreases on $(-\infty, 0]$ and increases on $[0, \infty)$

Theorem 1.4

Suppose that f is differentiable on (a, b)

- *If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing*
- *If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing*



Example 1.5

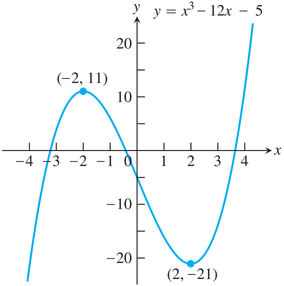
Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and decreasing

The function f is everywhere continuous and differentiable. The first derivative

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x + 2)(x - 2)$$

is zero at $x = -2$ and $x = 2$. These critical points subdivided the domain of f into intervals $(-\infty, -2), (-2, 2)$ and $(2, \infty)$ on which f' is either positive or negative. We determine the sign of f' by evaluating f at a convenient point in each subinterval.

	-2		2	
$3(x+2)$	$-$	• $+$		$+$
$x-2$	$-$	$-$	•	$+$
$f'(x)$	$+$	• $-$	•	$+$



Note: At the points where f has a minimum value, $f' < 0$ immediately to the left and $f' > 0$ immediately to the right. Thus the function is decreasing on the left of the minimum value and it is increasing on its right. Similarly, at the points where f has a maximum value, $f' > 0$ immediately to the left and $f' < 0$ immediately to the right. Thus the function is increasing on the left of the maximum value and decreasing on its right.

Theorem 1.6

First Derivative test for local maximums and minimums: Suppose that c is a critical point of f,

- *if f' changes from negative to positive at c, then f has a local minimum.*
- *if f' changes from positive to negative at c, then f has a local maximum.*
- *if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no maximum or minimum at c.*

Example 1.7

Find the critical points of

$$f(x) = x^{1/3}(x-4)$$

Identify the local maxima and minima.

We can write $f(x) = x^{1/3}(x-4) = x^{4/3} - 4x^{1/3}$.
The first derivative

$$f'(x) = \frac{d}{dx}(x^{4/3} - 4x^{1/3}) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x-1) = \frac{4(x-1)}{3x^{2/3}}$$

is zero at $x = 1$ and undefined at $x = 0$. So these are the critical points.

	0		1	
$4(x-1)$	$-$	$-$	$\bullet$	$+$
$3x^{2/3}$	$+$	$+$		$+$
$f'(x)$	$-$	$-$	$\bullet$	$+$