MA140-Engineering Calculus

Lecture 11

October 9, 2019

Example 1.1

Find dy/dx for

$$y = \tan^3[\sin^2(x^4)]$$

$y = u^3$
$u = \tan[\sin^2(x^4)] = \tan(v)$
$v = \sin^2(x^4) = t^2$
$t = \sin(x^4) = \sin(r)$
$r = x^4$

By the chain rule :

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dt} \cdot \frac{dt}{dr} \cdot \frac{dr}{dx}$$

$$\frac{dy}{dx} = (3u^2)(sec^2(v))(2t)(\cos r)(4x^3)$$

$$\frac{dy}{dx} = (3(\tan^2[\sin^2(x^4)])) \cdot (\sec^2(\sin^2(x^4))) \cdot (2\sin(x^4)) \cdot (\cos x^4) \cdot (4x^3)$$

Differentiation of Inverse functions

It is often useful to be able to express the derivative of an inverse function in terms of the derivatives of f.

Definition 1.2

If $y = f^{-1}(x)$, then $x = f(y)$ and also

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{f'(y)}$$

Example 1.3

Let us use the inverse rule to find $\frac{dy}{dx}$, when $y = x^{1/3}$

Note: We know that the answer is $\frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{\frac{-2}{3}}$, so we do not have to use the inverse rule but here we aim to use this rule to differentiate the function.

If $y = x^{\frac{1}{3}}$, then $y^3 = x$, or $x = y^3$, so

$$\frac{dx}{dy} = 3y^2$$

By the inverse rule:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{3y^2}$$

But $y = x^{\frac{1}{3}}$ so

$$\frac{dy}{dx} = \frac{1}{3(x^{\frac{1}{3}})^2} = \frac{1}{3}x^{-\frac{2}{3}}$$

Example 1.4

Find dy/dx for

$$y = \sin^{-1} x$$

Let $y = \sin^{-1} x$, then $x = \sin y \ (\star)$, so

$$\frac{dx}{dy} = \cos y \ \ (\star\star)$$

As $\sin^2 y + \cos^2 y = 1$, then $\cos y = \sqrt{1 - \sin^2 y}$, so by using $(\star)$

$$\cos y = \sqrt{1 - x^2}$$

Now using the inverse rule and from $(\star\star)$, we have

$$\frac{dy}{dx} = \frac{1}{\cos y} \quad or \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

Exercises

- Show that:

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

- Using the identity $1 + \tan^2 A = \sec^2 A$, show that:

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

- Find

$$\frac{d}{dx}[\tan^{-1}(e^{x^2})]$$

We had

$$\frac{d}{dx}(e^x) = e^x$$

We will find $\frac{dy}{dx}$ when $y = \ln(x)$
If $y = \ln(x)$, then $x = e^y$, so $\frac{dx}{dy} = e^y$.
Using the inverse rule we get

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{e^y} = \frac{1}{x}$$

Therefore

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

A function $f(x)$ can be written in a unique way as the sum of one even function and one odd function. The decomposition is

$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{even part}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{odd part}}.$$

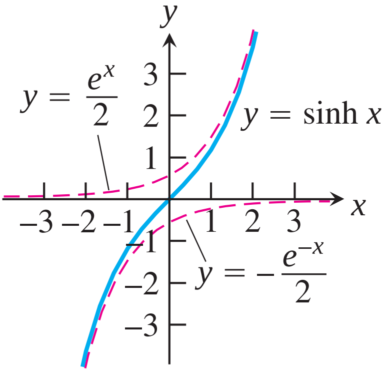
$$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{\text{even part}} + \underbrace{\frac{e^x - e^{-x}}{2}}_{\text{odd part}}.$$

Hyperbolic Functions

The hyperbolic functions are defined as follows

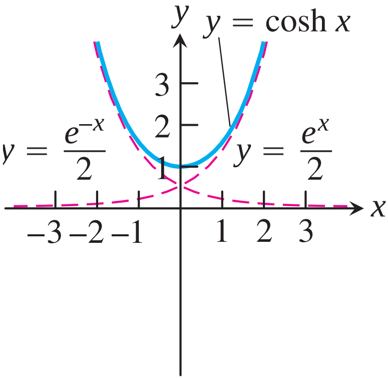
- *Hyperbolic sine of x*

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$



- *Hyperbolic cosine of x*

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

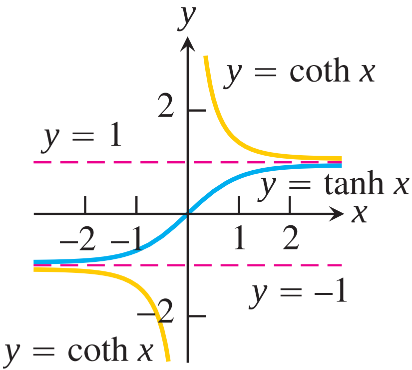


- *Hyperbolic tangent*

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

- *Hyperbolic cotangent*

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



Example 1.5

Find

$$\frac{d\sinh(x)}{dx}$$

We know that

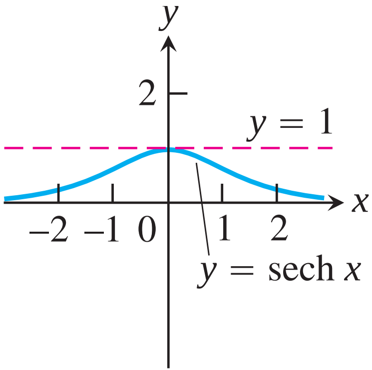
$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

So

$$\frac{d\sinh(x)}{dx} = \frac{d}{dx}(\frac{e^x - e^{-x}}{2}) = \frac{1}{2}\frac{d(e^x - e^{-x})}{dx} = \frac{1}{2}[e^x + e^{-x}] = \cosh(x)$$

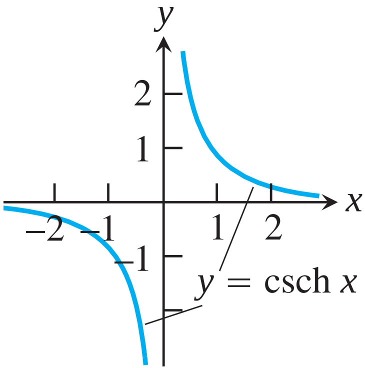
- *Hyperbolic secant*

$$\text{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}$$



- *Hyperbolic cosecant*

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}$$



Exercises

- Show that
$$\frac{d}{dx}(\cosh(x)) = \sinh(x)$$
- Show that
$$\frac{d}{dx}(\tanh(x)) = \operatorname{sech}^2(x)$$