Definition 1.3

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

In another notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 1.4

If $y = (x^3 + 4x^4 + 7)^{99}$, find $\frac{dy}{dx}$

Let $u = x^3 + 4x^4 + 7$, we can write y as $y = u^{99}$, then by chain rule we have:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 99u^{98}[3x^2 + 16x^3],$$

so

$$\frac{dy}{dx} = 99(x^3 + 4x^4 + 7)^{98}(3x^2 + 16x^3)$$

Example 1.5

If $y = \dfrac{1000}{(x^4 + 2x^2 + 8)^{40}}$, find $\frac{dy}{dx}$

$y = 1000(x^4 + 2x^2 + 8)^{-40}$.
Let $u = x^4 + 2x^2 + 8$, so we can write $y = 1000u^{-40}$
The Chain rule is:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Now:

$$\frac{dy}{du} = (1000)(-40)u^{-41}$$

$$\frac{du}{dx} = 4x^3 + 4x$$

so

$$\frac{dy}{dx} = -40000u^{-41}[4x^3 + 4x]$$

then

$$\frac{dy}{dx} = \frac{-40000}{u^{41}}(4x^3 + 4x)$$

or

$$\frac{dy}{dx} = \frac{-40000(4x^3 + 4x)}{(x^4 + 2x^2 + 8)^{41}}$$

Note: Sometimes it is useful to involve a second (or more) intermediate function

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

Example 1.6

Find $\frac{dy}{dx}$, when

$$y = \sin^4(x^5 + 7)$$

Let $u = \sin(x^5 + 7)$ and let $v = x^5 + 7$
so the chain rule gives

$$\frac{d\sin^4(x^5+7)}{dx} = \frac{d\sin^4(x^5+7)}{d\sin(x^5+7)} \cdot \frac{d\sin(x^5+7)}{d(x^5+7)} \cdot \frac{d(x^5+7)}{dx}$$

$$\left[4\sin^3(x^5+7)\right] \cdot \left[\cos(x^5+7)\right] \cdot \left[5x^4\right] = 20x^4 \cdot \sin^3(x^5+7) \cdot \cos(x^5+7)$$

Exercises

Find $\frac{dy}{dx}$, if:

- $$y = x^2 e^{\sin x}$$
- $$y = \tan^3[\sin^2(x^4)]$$