

# MA140-Engineering Calculus

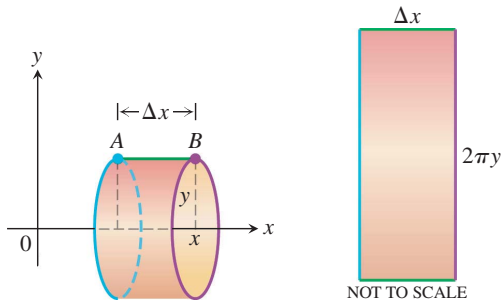
## Lecture 30

November 10, 2017

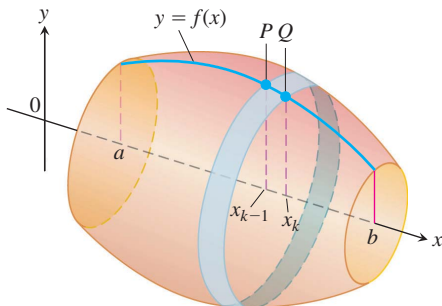
We begin by rotating the line segment  $y = r$ , from  $x = A$  to  $x = B$ .  
 If we rotate this line segment  $AB$  having length  $\Delta x$  about the  $x$ -axis, we generate a cylinder with surface area

$$2\pi y\Delta x = 2\pi r\Delta x$$

This area is the same as that of a rectangle with side lengths  $\Delta x$  and  $2\pi y$



**Surface area:** We are interested in the surface generated by rotating the curve about the  $x$ -axis.



Suppose that the arc length from  $p$  to  $Q$  is  $\Delta S_k$ , so the surface area of the typical band above is:

$$\Delta S_k 2\pi f(x_k)$$

So the surface area can be approximated by the following sum:

$$S \cong \sum_{k=1}^n \Delta S_k 2\pi f(x_k)$$

From the last lecture, we know that

$$\Delta S_k = \Delta x_k \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}$$

So:

$$S \cong \sum_{k=1}^n (\Delta x_k \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}) 2\pi f(x_k)$$

Let  $n \rightarrow \infty$  or  $\Delta x_k \rightarrow 0$ , then:

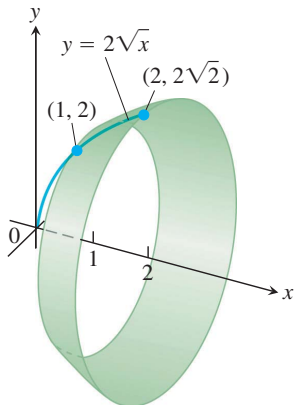
$$S = \lim_{n \rightarrow \infty} \sum_{k=1}^n (\Delta x_k \sqrt{1 + (\frac{\Delta y_k}{\Delta x_k})^2}) 2\pi f(x_k)$$

As this is a Riemann sum:

$$S = 2\pi \int_a^b f(x) \sqrt{1 + (\frac{dy}{dx})^2} dx$$

## Example 1.1

Find the area of the surface generated by revolving the curve  $y = 2\sqrt{x}$ ,  $1 \leq x \leq 2$ , about the  $x$ -axis



We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

with:

$$a = 1, b = 2, y = 2\sqrt{x}, \quad \frac{dy}{dx} = \frac{1}{\sqrt{x}}$$

So:

$$\begin{aligned} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2} \\ &= \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}} \end{aligned}$$

So:

$$\begin{aligned} S &= \int_1^2 2\pi \cdot 2\sqrt{x} \cdot \frac{\sqrt{x+1}}{\sqrt{x}} dx = 4\pi \int_1^2 \sqrt{x+1} dx \\ &= 4\pi \cdot \frac{2}{3} (x+1)^{3/2} \Big|_1^2 = \frac{8\pi}{3} (3\sqrt{3} - 2\sqrt{2}) \end{aligned}$$

### Example 1.2

A reflector is formed by rotating  $y = \sqrt{x}$  between  $x = 0$  and  $x = 1$  about the  $x$ -axis. What is the surface area.

### Example 1.3

Find the area of the surface generated by revolving the curve  $y = \frac{x^3}{9}$  between  $x = 0$  and  $x = 2$

We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

with:

$$a = 0, b = 2, y = \frac{x^3}{9}, \quad \frac{dy}{dx} = \frac{x^2}{3}$$

So:

$$\begin{aligned} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \sqrt{1 + \left(\frac{x^2}{3}\right)^2} \\ &= \sqrt{1 + \frac{x^4}{9}} = \sqrt{\frac{9 + x^4}{9}} = \frac{\sqrt{9 + x^4}}{3} \end{aligned}$$

So:

$$\begin{aligned} S &= \int_0^2 2\pi \cdot \left(\frac{x^3}{9}\right) \cdot \frac{\sqrt{9+x^4}}{3} dx = \frac{2\pi}{27} \int_0^2 x^3 \sqrt{9+x^4} dx \\ &= \frac{2\pi}{27} \cdot \frac{(9+x^4)^{3/2}}{6} \Big|_0^2 = \frac{98\pi}{77} \end{aligned}$$