MA140-Engineering Calculus

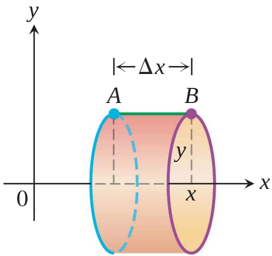
Lecture 30

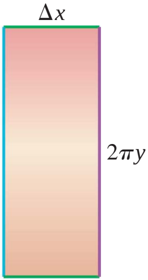
November 10, 2017

We begin by rotating the line segment $y = r$, from $x = A$ to $x = B$.
If we rotate this line segment AB having length Δx about the $x-$axis, we generate a cylinder with surface area

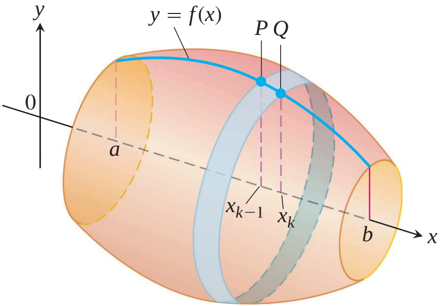
$$2\pi y \Delta x = 2\pi r \Delta x$$

This area is the same as that of a rectangle with side lengths Δx and $2\pi y$





Surface area: We are interested in the surface generated by rotating the curve about the x-axis.



Suppose that the arc length from p to Q is ΔS_k, so the surface area of the typical band above is:

$$\Delta S_k 2\pi f(x_k)$$

So the surface area can be approximated by the following sum:

$$S \cong \sum_{k=1}^{n} \Delta S_k 2\pi f(x_k)$$

From the last lecture, we know that

$$\Delta S_k = \Delta x_k \sqrt{1 + (\frac{\Delta y_k}{\Delta x_k})^2}$$

So:

$$S \cong \sum_{k=1}^{n} (\Delta x_k \sqrt{1 + (\frac{\Delta y_k}{\Delta x_k})^2}) 2\pi f(x_k)$$

Let $n \to \infty$ or $\Delta x_k \to 0$, then:

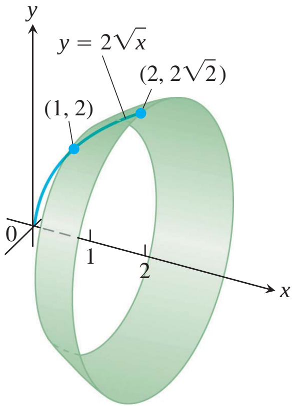
$$S = \lim_{n\to\infty} \sum_{k=1}^{n} (\Delta x_k \sqrt{1 + (\frac{\Delta y_k}{\Delta x_k})^2}) 2\pi f(x_k)$$

As this is a Riemann sum:

$$S = 2\pi \int_a^b f(x) \sqrt{1 + (\frac{dy}{dx})^2} dx$$

Example 1.1

Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$, about the $x-$axis



We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{1 + (\frac{dy}{dx})^2}$$

with:

$$a = 1, b = 2, y = 2\sqrt{x}, \quad \frac{dy}{dx} = \frac{1}{\sqrt{x}}$$

So:

$$\sqrt{1 + (\frac{dy}{dx})^2} = \sqrt{1 + (\frac{1}{\sqrt{x}})^2}$$

$$= \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}}$$

So:

$$S = \int_1^2 2\pi \cdot 2\sqrt{x} \cdot \frac{\sqrt{x+1}}{\sqrt{x}} dx = 4\pi \int_1^2 \sqrt{x+1} dx$$

$$= 4\pi \cdot \frac{2}{3}(x+1)^{3/2}\Big]_1^2 = \frac{8\pi}{3}(3\sqrt{3} - 2\sqrt{2})$$

Example 1.2

A reflector is formed by rotating $y = \sqrt{x}$ between $x = 0$ and $x = 1$ about the $x-$axis. What is the surface area.

Example 1.3

Find the area of the surface generated by revolving the curve $y = \frac{x^3}{9}$ between $x = 0$ and $x = 2$

We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{1 + (\frac{dy}{dx})^2}$$

with:

$$a = 0, b = 2, y = \frac{x^3}{9}, \quad \frac{dy}{dx} = \frac{x^2}{3}$$

So:

$$\sqrt{1 + (\frac{dy}{dx})^2} = \sqrt{1 + (\frac{x^2}{3})^2}$$

$$= \sqrt{1 + \frac{x^4}{9}} = \sqrt{\frac{9 + x^4}{9}} = \frac{\sqrt{9 + x^4}}{3}$$

So:

$$S = \int_0^2 2\pi \cdot (\frac{x^3}{9}) \cdot \frac{\sqrt{9+x^4}}{3} dx = \frac{2\pi}{27} \int_0^2 x^3 \sqrt{9+x^4} dx$$

$$= \frac{2\pi}{27} \cdot \frac{(9+x^4)^{3/2}}{6} \Big]_0^2 = \frac{98\pi}{77}$$