MA140-Engineering Calculus

Lecture 34

November 21, 2017

Differential Equations:

Definition 1.1

An equation involving a derivative is a differential equation. For example

$$\frac{dy}{dt} = ky \ \ (\star)$$

where k is some constant and y some function of t.

Are there any solutions to the above equation?

Consider for instance

$$y = e^{kt}$$

Then:

$$\frac{dy}{dt} = ke^{kt} = ky$$

So $y = e^{kt}$ is a solution of $(\star)$, also $y = 5e^{kt}$ satisfies the equation so we see that

$$y = Ae^{kt}$$

is a solution to $(\star)$ for any constant A

World Population:

Example 1.2

Suppose that the world population in 1960 was 3.06 billion, also suppose during this period the world population increased by 2% per year. Calculate today's population.

Strategy: Start with a simple model and modify it if necessary
Let $y(t) =$ world population at time t, measured in years.

Malthusian Law:

In 1798 English economist Thomas Malthus suggested that the rate of change of a population is proportional to the size of the population.

Definition 1.3

Malthusian Law:

$$\frac{dy}{dt} = ky$$

This means that at time t the world population is

$$y = Ae^{kt}$$

for some constants A, k

Let's start time $t = 0$ in 1960, so

$$y(0) = 3.06 \ \ billion = Ae^{0k} = A \Rightarrow A = 3.06 \ \ billion$$

$$y(1) = 1.02y(0)$$

$$y(2) = 1.02y(1) = (1.02)^2 y(0)$$

$$y(3) = 1.02y(2) = (1.02)^3 y(0)$$

So in general:

$$y(t) = (1.02)^t y(0) \Rightarrow \frac{y(t)}{y(0)} = (1.02)^t = \frac{Ae^{kt}}{A} = e^{kt}$$

So $e^k = 1.02$, therefore:

$$\ln(e^k) = \ln(1.02) \Rightarrow k \ln(e) = \ln(1.02) \Rightarrow k = \ln(1.02) = 0.0198$$

So roughly

$$k = 0.02$$

Therefore:

$$y(t) = 3.06e^{0.02t}$$

Now let's calculate today's population using our model.

$$t = 2017 - 1960 = 57$$

So:

$$y = 3.06e^{0.02\times 57} = 9.38 \ \ billion$$

Definition 1.4

A **differential equation** is an equation containing derivatives (we often write **D.E.** for differential equation). For example:

- $\frac{dy}{dx} = 2xy$
- $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 4y = e^x$

Definition 1.5

The **order of a D.E.** is the order of the highest derivative in the D.E. For example:

- $\frac{dy}{dx} = 2xy$ is a first order differential equation
- $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 4y = e^x$ is a second order differential equation

Separable Differential Equations:

Definition 1.6

A first order D.E. of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is called a **separable D.E.**
(where $g(x)$ is a function of x and $h(y)$ is a function of y).

This D.E. can be solved as follows:

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x) \Rightarrow \int \frac{1}{h(y)}dy = \int g(x)dx$$

Then integrate and solve for y if possible.

Example 1.7

Solve the separable D.E.

$$\frac{dy}{dx} = \frac{3x^2}{\sin y}$$

$$\sin y dy = 3x^2 dx \Rightarrow \int \sin y dy = 3 \int x^2 dx$$

$$\Rightarrow -\cos y = x^3 + C \Rightarrow \cos y = -x^3 - C$$

$$\Rightarrow \cos^{-1}(\cos y) = \cos^{-1}(-x^3 - C) \Rightarrow y = \cos^{-1}(-x^3 - C)$$

Example 1.8

Solve the separable D.E.

$$\frac{dy}{dx} = 1 + y^2$$

Given that when $x = 0$ then $y = 0$

$$\frac{1}{1+y^2}\frac{dy}{dx} = 1 \Rightarrow \int \frac{1}{1+y^2}dy = \int 1dx$$

Let $y = \tan\theta$ so $dy = \sec^2\theta d\theta$. Therefore:

$$\int \frac{1}{1+\tan^2\theta}\sec^2\theta d\theta = \int dx$$

$$\Rightarrow \int d\theta = \int dx$$

So:

$$\theta = x + c$$

Now $x = 0, y = 0 \Rightarrow 0 = \tan\theta \Rightarrow \theta = 0$, thus

$$0 = 0 + c \Rightarrow c = 0$$

Therefore $\theta = x$ and $y = \tan x$ is our solution.

First Order Linear Differential Equations (The Integrating Factor Method):

Definition 1.9

A D.E. of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is called a **first order linear D.E.**, where $P(x)$ and $Q(x)$ are functions of x.

We can find the general solution of this D.E. as follows:

1.) Find the **integrating factor**

$$e^{\int P(x)dx}$$

2.) Multiply the D.E. by the integrating factor (I.F.) to get:

$$e^{\int P(x)dx}\left(\frac{dy}{dx} + P(x)y\right) = e^{\int P(x)dx}Q(x) \quad (\star)$$

3.) Note that the L.H.S. of ($\star$) equals:

$$\frac{d}{dx}(ye^{\int P(x)dx})$$

So ($\star$) becomes:

$$d(ye^{\int P(x)dx}) = e^{\int P(x)dx}Q(x)dx$$

4.) Integrate both sides and solve for y.
This algorithm is called the **integrating factor method**.

Example 1.10

Solve the first order linear differential equation

$$\frac{dy}{dx} - 3y = 0$$

using the integrating factor method.

1.)

$$I.F. = e^{\int P(x)dx} = e^{\int -3dx} = e^{-3x}$$

(note that we do not include the arbitrary constant C).

2.) Multiply the D.E. by the I.F. to get

$$e^{-3x}(\frac{dy}{dx} - 3y) = 0e^{-3x} \quad (\star)$$

3.) Recall that the L.H.S. of $(\star)$ equals

$$\frac{d}{dx}(ye^{-3x})$$

therefore

$$\frac{d}{dx}(ye^{-3x}) = 0 \Rightarrow d(ye^{-3x}) = 0dx$$

4.) Integrate and solve for y:

$$\int d(ye^{-3x}) = \int 0dx \Rightarrow ye^{-3x} = 0 + C \Rightarrow y = Ce^{3x}$$