MA140-Engineering Calculus

Lecture 7

September 22, 2017

Example 1.1

Prove formally that

$$\lim_{x\to 7}\frac{-x^2+9x-14}{x-7}=-5$$

Given ϵ, then

$$|f(x)-l|<\epsilon \Leftrightarrow |\frac{-x^2+9x-14}{x-7}-(-5)|<\epsilon$$

$$\Leftrightarrow |\frac{-(x-7)(x-2)}{x-7}+5|<\epsilon$$

$$\Leftrightarrow |-(x-2)+5|<\epsilon$$

$$\Leftrightarrow |-x+7|<\epsilon \Leftrightarrow |x-7|<\epsilon$$

so we should pick $\epsilon=\delta$

- $|x - a| < \delta$ is the same as $a - \delta < x < a + \delta$ or $-\delta < x - a < \delta$

Example 1.2

solve
$$|x - 7| < 3$$

$$-3 < x - 7 < 3 \Leftrightarrow 7 - 3 < x < 7 + 3 \Leftrightarrow 4 < x < 10$$

- For $x \in \mathbb{R}$:
$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

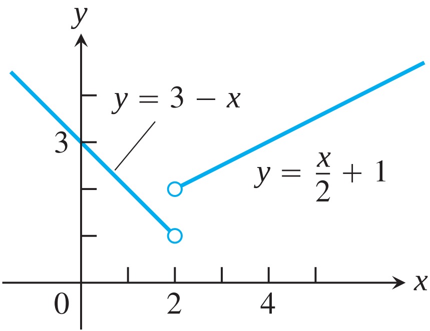
- When multiplying an inequality by a negative number flip the inequality sign.

One-sided limits

Example 1.3

Let

$$f(x) = \begin{cases} 3 - x, & x < 2 \\ \frac{x}{2} + 1, & x > 2 \end{cases}$$



Note:

- The function approaches 1 as x approaches 2 from the left.
- The function approaches 2 as x approaches 2 from the right.

Notation:

$$\lim_{x\to 2^-} f(x) = 1 \quad \lim_{x\to 2^+} f(x) = 2$$

These one-sided limits can be defined formally using the ϵ/δ notation.
Clearly: $\lim_{x\to 2} f(x)$ above does not exist.

Theorem 1.4

A function $f(x)$ has a limit as x approaches a if and only if it has left-handed and right-handed limits there and these one-sided limits are equal.

$$\lim_{x\to a} f(x) = l \quad \Leftrightarrow \quad \lim_{x\to a^-} f(x) = l \quad \textit{and} \quad \lim_{x\to a^+} f(x) = l$$

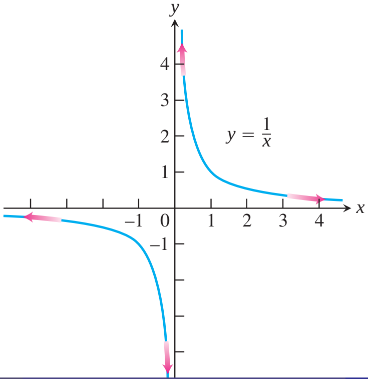
Example 1.5

$$\lim_{x\to 0^+} \sqrt{x} = 0$$

Limits at Infinity

Example 1.6

Find the limit of $y = \frac{1}{x}$ as $x \to \pm\infty$



Asymptotes

Example 1.7

$$\lim_{x\to\infty} \frac{x+3}{x+2}$$

When dealing with limits at infinity of rational functions it can be useful to divide top and bottom by the highest power of the bottom.

$$\lim_{x\to\infty} \frac{x+3}{x+2} = \lim_{x\to\infty} \frac{x(1+\frac{3}{x})}{x(1+\frac{2}{x})} = \lim_{x\to\infty} \frac{(1+\frac{3}{x})}{(1+\frac{2}{x})} = 1$$

this exercise tells us that when x get very big, the function tends to 1.

We say that the function $\dfrac{x+3}{x+2}$ has a *horizontal asymptote* $y = 1$

Definition 1.8

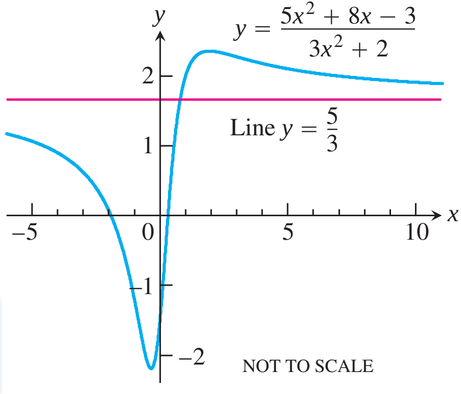
A line $y = b$ is a horizontal asymptote of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad or \quad \lim_{x\to-\infty} f(x) = b$$

Example 1.9

Find the horizontal asymptote of $f(x) = \frac{5x^2+8x-3}{3x^2+2}$

$$\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to\infty} \frac{5+(8/x)-(3/x^2)}{3+(2/x^2)} = \frac{5+0-0}{3+0} = \frac{5}{3}$$



Note that the denominator of $\dfrac{x+3}{x+2}$ is zero when $x=-2$.

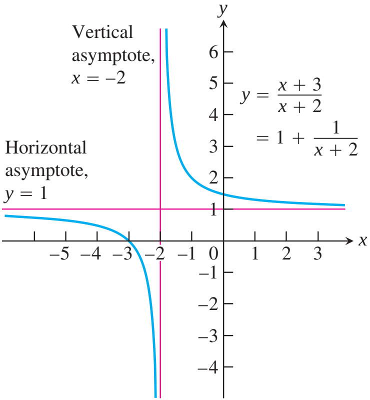
$$\lim_{x\to -2^-} \frac{x+3}{x+2} = -\infty \qquad \lim_{x\to -2^+} \frac{x+3}{x+2} = \infty$$

We say that the function $\dfrac{x+3}{x+2}$ has a *vertical asymptote* $x=-2$

Definition 1.10

A line $x=a$ is a vertical asymptote of the graph of a function $y=f(x)$ if either

$$\lim_{x\to a^+} f(x) = \infty \quad \textit{or} \quad \lim_{x\to a^-} f(x) = \infty$$


y
Vertical
asymptote,
x = –2
6
5
4
3
2
1
$y = \frac{x+3}{x+2}$
$= 1 + \frac{1}{x+2}$
Horizontal
asymptote,
y = 1
–5 –4 –3 –2 –1 0 1 2 3
x
–1
–2
–3
–4

Example 1.11

Find all asymptotes of $f(x)$ and plot

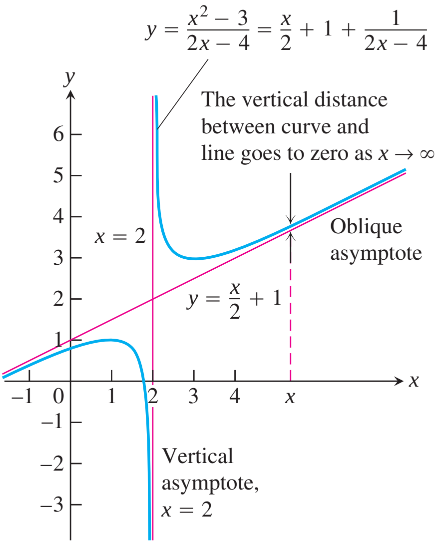
$$f(x) = \frac{x^2 - 3}{2x - 4}$$

writing this as a strictly proper rational function

$$f(x) = \frac{x}{2} + 1 + \frac{1}{2x - 4}$$

so, when x gets very big $f(x)$ tends to $\frac{x}{2} + 1$.
Also when x gets very negative, $f(x)$ tends to $\frac{x}{2} + 1$.
when $2x - 4 = 0$ or $x = 2$, we have a vertical asymptote.
the zeros of the function are:

$$x^2 - 3 = 0 \Rightarrow x = \pm 3$$



$y = \frac{x^2 - 3}{2x - 4} = \frac{x}{2} + 1 + \frac{1}{2x - 4}$
The vertical distance between curve and line goes to zero as $x \to \infty$
Oblique asymptote
$y = \frac{x}{2} + 1$
$x = 2$
Vertical asymptote, $x = 2$
y
x
6
5
4
3
2
1
0
–1
–2
–3
1
2
3
4

Example 1.12

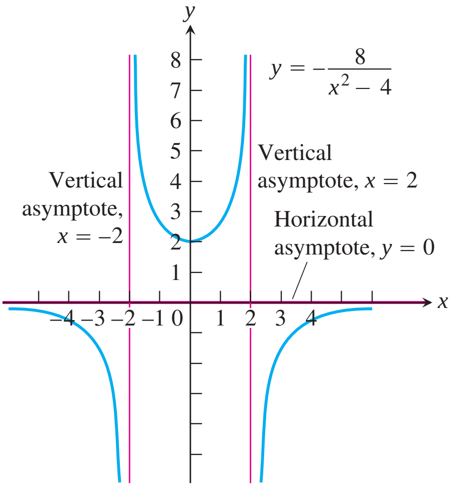
Find the horizontal and vertical asymptotes of the graph of

$$f(x) = -\frac{8}{x^2 - 4}$$

First, since $\lim_{x\to\infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote. Also, since

$$\lim_{x\to 2^+} f(x) = -\infty \quad and \quad \lim_{x\to 2^-} f(x) = \infty$$

the line $x = 2$ is a vertical asymptote both from the right and from the left.


y
x
8
7
6
5
4
3
2
1
−4 −3 −2 −1 0 1 2 3 4
$y = -\frac{8}{x^2 - 4}$
Vertical asymptote, $x = 2$
Vertical asymptote, $x = -2$
Horizontal asymptote, $y = 0$

Exercises

- $$\lim_{x \to 2^-} \frac{x-3}{x^2-4}$$
- $$\lim_{x \to 2^-} \frac{x-3}{x^2-4}$$
- $$\lim_{x \to 2} \frac{2-x}{(x-2)^3}$$