

MA140-Engineering Calculus

Lecture 6

September 14, 2017

Example 1.1

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$$

Note: $\cos(2A) = 1 - 2 \sin^2(A)$

Let: $A = x/2$, then $\Rightarrow \cos x = 1 - 2 \sin^2 x/2$, so

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} &= \lim_{x \rightarrow 0} \frac{1 - [1 - 2 \sin^2(x/2)]}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{2 \sin^2(x/2)}{x^2} = \lim_{x \rightarrow 0} 2 \frac{\sin x/2}{x/2} \cdot \frac{\sin x/2}{x/2} \cdot (1/2)(1/2) \\ &= \lim_{x \rightarrow 0} 2 \frac{\sin x/2}{x/2} \cdot \frac{\sin x/2}{x/2} \cdot (1/2)(1/2) \\ &= \lim_{x \rightarrow 0} \frac{\sin x/2}{x/2} \cdot \lim_{x \rightarrow 0} \frac{\sin x/2}{x/2} \cdot 1/2 \\ &= (1)(1)(1/2) = 1/2 \end{aligned}$$

Useful approximation:

1. when $x \rightarrow 0$ then $\cos x \approx 1 - \frac{x^2}{2}$
2. when $x \rightarrow 0$ then $\sin x \approx x - \frac{x^3}{6}$

Example 1.2

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3}$$

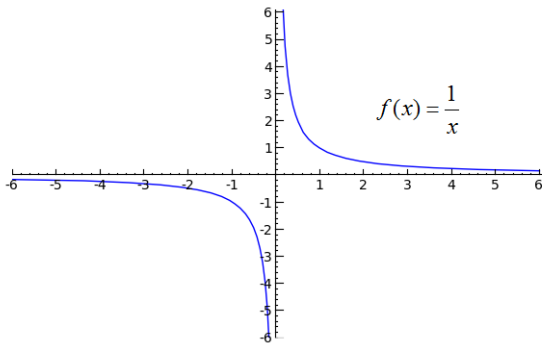
$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin x - x}{x^3} &= \lim_{x \rightarrow 0} \frac{\left[x - \frac{x^3}{6}\right] - x}{x^3} \\ &= \lim_{x \rightarrow 0} \frac{-\frac{x^3}{6}}{x^3} = -1/6 \end{aligned}$$

a lot of limit analysis ends with ∞ , in these cases we say that the limit does not exist.

$$\frac{\textit{infinity}}{\textit{constant} \neq 0} \quad \textit{or} \quad \frac{\textit{constant} \neq 0}{0} \Rightarrow \infty$$

Example 1.3

$$\lim_{x \rightarrow 0} 1/x = 1/0 = \infty$$



Example 1.4

If $f(x) = \frac{x+1}{x^3}$, find $\lim_{x \rightarrow 0} f(x)$

$f(x)$ tends to $\frac{\text{constant} \approx 1}{\approx 0}$
 so $\lim_{x \rightarrow 0} f(x) = \infty$, it does not exist.

Note:

$$\frac{\text{constant} \neq 0}{\text{infinitly}} \quad \text{or} \quad \frac{0}{\text{constant} \neq 0} \Rightarrow 0$$

one can think of sequences as values of continuous functions at $x = 1, 2, 3, \dots$

for example $f(x) = x^2 + 1$, $a_n = n^2 + 1$

$\{a_n\} = \{2, 5, 10, 17, \dots\}$

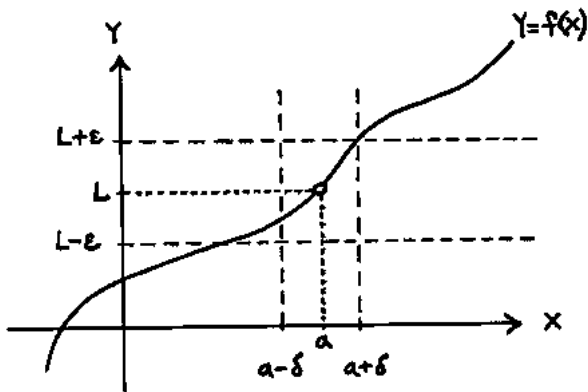
we can use our limit theory on continuous functions to decide if a sequence has a limit.

Example 1.5

Does $\{a_n\}$ with $a_n = \frac{n^2+1}{n+1}$ have a limit.

$$\lim_{n \rightarrow \infty} \frac{n^2+1}{n+1} = \lim_{n \rightarrow \infty} \frac{n+\frac{1}{n}}{1+\frac{1}{n}} = \infty$$

$\{a_n\}$ has no limit if diverges to infinity



- a is a fixed number which x can take i.e. $x = a$
- δ and ϵ are small positive numbers i.e. $\delta > 0$
- $|x - a| < \delta$ the distance from x to a is less than δ
- $f(x)$ approaches a number l as x approaches a

So if I am able to find a δ given an ϵ then (in an informal way) this means: that we can make the value of $f(x)$ as close as we like to L by taking x sufficiently close to a .

Formally:

A function $f(x)$ is said to approach a limit l as x approaches the value a , if given any small positive number ϵ , it is possible to find a positive number δ , such that for any x :

$$|x - a| < \delta \quad \Rightarrow \quad |f(x) - l| < \epsilon$$

we write

$$\lim_{x \rightarrow a} f(x) = l$$

Example 1.6

Prove formally that:

$$\lim_{x \rightarrow 3} (4x - 5) = 7$$

Given ϵ , then

$$|f(x) - l| < \epsilon \Leftrightarrow |(4x - 5) - 7| < \epsilon$$

$$\Leftrightarrow |4x - 5 - 7| < \epsilon$$

$$\Leftrightarrow |4x - 12| < \epsilon$$

$$\Leftrightarrow 4|x - 3| < \epsilon$$

$$\Leftrightarrow |x - 3| < \epsilon/4$$

so if we pick $\delta = \epsilon/4$ then from above we see that if $|x - 3| < \delta = \epsilon/4$ implies δ so $|(4x - 5) - 7| < \epsilon$.

Exercises

(1)

$$\lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 - 4}$$

(2)

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{(\sqrt{x} - 1)(x + 2)}$$

(3)

$$\lim_{x \rightarrow 0} \frac{\sin^2 x}{\sin 2x \cdot \tan 4x}$$

(4)

$$\lim_{x \rightarrow 0} x \cos \frac{1}{x^2}$$