MA140-Engineering Calculus

Lecture 5

September 14, 2017

Example 1.1

Evaluate:

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}$$

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{(x-1)(x+2)}{x(x-1)} = \lim_{x \to 1} \frac{x+2}{x} = 3$$

Note: If the denominator is zero, canceling common factors in the numerator and denominator may reduce the fraction to one whose denominator is no longer zero

Example 1.2

$$\lim_{x\to 2}\left(\frac{1}{2}-\frac{1}{x}\right)\frac{1}{x-2}$$

$$\begin{aligned}\lim_{x\to 2}\left(\frac{1}{2}-\frac{1}{x}\right)\frac{1}{x-2} &= \lim_{x\to 2}\frac{x-2}{2x}\cdot\frac{1}{x-2}\\ &= \lim_{x\to 2}\frac{1}{2x} = \frac{1}{2(2)} = \frac{1}{4}\end{aligned}$$

Example 1.3

Evaluate

$$\lim_{x\to 0} \frac{\sqrt{1+x^2}-1}{x^2}$$

$$\lim_{x\to 0} \frac{\sqrt{1+x^2}-1}{x^2} = \lim_{x\to 0} \frac{(\sqrt{1+x^2}-1)(\sqrt{1+x^2}+1)}{x^2(\sqrt{1+x^2}+1)}$$

$$\lim_{x\to 0} \frac{(1+x^2-1)}{x^2(\sqrt{1+x^2}+1)} = \lim_{x\to 0} \frac{(x^2)}{x^2(\sqrt{1+x^2}+1)}$$

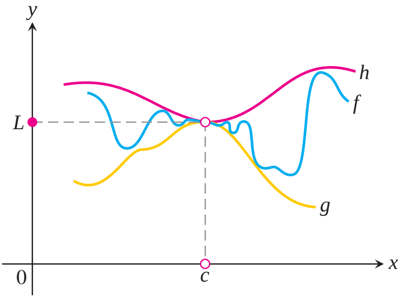
$$\lim_{x\to 0} \frac{1}{\sqrt{1+x^2}+1} = \frac{1}{\sqrt{1}+1} = 1/2$$

Theorem 1.4

Sandwich Theorem Suppose that $g(x) \leq f(x) \leq h(x)$ *for all* x *in some open interval containing* c*, except possibly at* c *itself. Suppose also that*

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$$

Then $\lim_{x \to c} f(x) = L$



Example 1.5

Given

$$1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2} \quad forall \quad x \neq 0$$

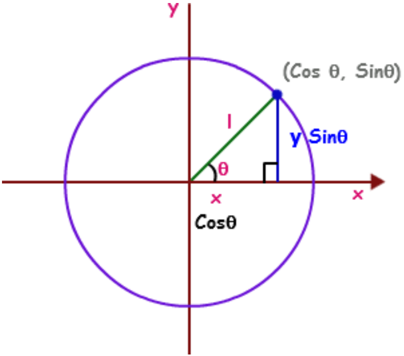
Find the $\lim_{x \to 0} u(x)$, no matter how complicated u is.

Since

$$\lim_{x \to 0} (1 - \frac{x^2}{4}) = 1 \quad and \quad \lim_{x \to 0} (1 + \frac{x^2}{2}) = 1$$

the Sandwich theorem implies that

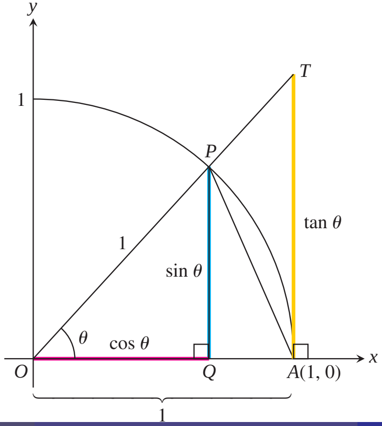
$$\lim_{x \to 0} u(x) = 1$$


y
(Cos θ, Sinθ)
l
y Sinθ
θ
x
Cosθ
x

Example 1.6

An important limit:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$



Area of $\triangle OAP <$ Area of the sector $OAP <$ Area of $\triangle OAT$
Area of $\triangle OAP = \frac{1}{2}\sin\theta$
Area of the sector $OAP = \frac{1}{2}r^2\theta = \frac{\theta}{2}$
Area of $\triangle OAT = \frac{1}{2}\tan\theta$

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

Taking the reciprocals reverses the inequalities so:

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

Since $\lim_{\theta\to 0}\cos\theta = \lim_{\theta\to 0} 1 = 1$, The Sandwich theorem gives:

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

Example 1.7

$$\lim_{x\to 0} \frac{\tan 3x}{\sin 2x}$$

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan 3x}{\sin 2x} &= \lim_{x\to 0} \frac{\sin 3x}{\cos 3x} \cdot \frac{1}{\sin 2x} \\
&= \lim_{x\to 0} \sin 3x \cdot \frac{1}{\cos 3x} \cdot \frac{1}{\sin 2x} \\
&= \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot \frac{1}{\cos 3x} \cdot \frac{2x}{\sin 2x} \cdot \frac{3x}{2x} \\
&= (\lim_{x\to 0} \frac{\sin 3x}{3x})(\lim_{x\to 0} \frac{1}{\cos 3x})(\lim_{x\to 0} \frac{2x}{\sin 2x})(\lim_{x\to 0} \frac{3x}{2x}) \\
&= (1)(1)(1)(3/2) = 3/2
\end{aligned}$$