

# MA140-Engineering Calculus

## Lecture 35

November 22, 2017

## First Order Linear Differential Equations (The Integrating Factor Method):

## Definition 1.1

A D.E. of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is called a **first order linear D.E.**, where  $P(x)$  and  $Q(x)$  are functions of  $x$ .

We can find the general solution of this D.E. as follows:

1.) Find the **integrating factor**

$$e^{\int P(x)dx}$$

2.) Multiply the D.E. by the integrating factor (I.F.) to get:

$$e^{\int P(x)dx} \left( \frac{dy}{dx} + P(x)y \right) = e^{\int P(x)dx} Q(x) \quad (\star)$$

3.) Note that the L.H.S. of (\*) equals:

$$\frac{d}{dx}(ye^{\int P(x)dx})$$

So (\*) becomes:

$$d(ye^{\int P(x)dx}) = e^{\int P(x)dx}Q(x)dx$$

4.) Integrate both sides and solve for  $y$ .

This algorithm is called the **integrating factor method**.

## Example 1.2

Solve the first order linear differential equation

$$\frac{dy}{dx} - 3y = 0$$

using the integrating factor method.

1.)

$$I.F. = e^{\int P(x)dx} = e^{\int -3dx} = e^{-3x}$$

(note that we do not include the arbitrary constant  $C$ ).

2.) Multiply the D.E. by the I.F. to get

$$e^{-3x} \left( \frac{dy}{dx} - 3y \right) = 0e^{-3x} \quad (\star)$$

3.) Recall that the L.H.S. of  $(\star)$  equals

$$\frac{d}{dx}(ye^{-3x})$$

therefore

$$\frac{d}{dx}(ye^{-3x}) = 0 \Rightarrow d(ye^{-3x}) = 0dx$$

4.) Integrate and solve for  $y$ :

$$\int d(ye^{-3x}) = \int 0dx \Rightarrow ye^{-3x} = 0 + C \Rightarrow y = Ce^{3x}$$

### Example 1.3

Solve the first order linear differential equation

$$\frac{dy}{dx} - y = e^x$$

using the integrating factor method.

1.)

$$I.F. = e^{\int P(x)dx} = e^{\int -1dx} = e^{-x}$$

(note that we do not include the arbitrary constant  $C$ ).

2.) Multiply the D.E. by the I.F. to get

$$e^{-x} \left( \frac{dy}{dx} - y \right) = e^x \cdot e^{-x} \quad (\star)$$

3.) Recall that the L.H.S. of  $(\star)$  equals

$$\frac{d}{dx}(ye^{-x})$$

therefore

$$\frac{d}{dx}(ye^{-x}) = 1 \Rightarrow d(ye^{-x}) = 1dx$$

4.) Integrate and solve for  $y$ :

$$\int d(ye^{-x}) = \int 1dx \Rightarrow ye^{-x} = x + C \Rightarrow y = xe^x + c$$

## Last year exam Solutions:

## Question 1.(a).i:

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x-1}{(\sqrt{x}-1)(x+2)} &= \lim_{x \rightarrow 1} \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(x+2)} \\ &= \lim_{x \rightarrow 1} \frac{\sqrt{x}+1}{x+2} = \frac{2}{3}\end{aligned}$$

## Question 1.(a).ii:

we apply L'Hôpital's Rule:

$$\lim_{\theta \rightarrow 0} \frac{6 \sin \theta}{\theta + 2 \tan \theta} \stackrel{\text{H}}{=} \lim_{\theta \rightarrow 0} \frac{6 \cos \theta}{1 + 2 \sec^2 \theta} = \frac{6}{3} = 2$$

## Question 2.(a).i:

The derivative of the function  $f(x)$  with respect to the variable  $x$  is the function  $f'$  or  $\frac{df}{dx}$  whose value at  $x$  is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

## Question 2.(a).ii:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^2 + (x+h) - (x^2 + x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + h^2 + 2xh + x + h - x^2 - x}{h} = \lim_{h \rightarrow 0} \frac{h^2 + 2xh + h}{h} \\ &= \lim_{h \rightarrow 0} (h + 2x + 1) = 2x + 1 \end{aligned}$$

## Question 2.(b).i:

$$f(x) = e^{\cos x^2} \sin x$$

$$\Rightarrow f'(x) = (e^{\cos x^2} \sin x)' = (e^{\cos x^2})' \sin x + e^{\cos x^2} (\sin x)' \quad (\star)$$

To differentiate  $e^{\cos x^2}$  we use the chain rule:

Let  $y = e^{\cos x^2} = e^u$ , where  $u = \cos x^2$ , then by chain rule we have:

$$y' = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot (\cos x^2)'$$

To differentiate  $\cos x^2$  we use the chain rule again:

$u = \cos x^2 = \cos v$ , where  $v = x^2$ , then by chain rule we have:

$$u' = \frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx} = (-\sin v)(2x) = (-\sin x^2)(2x)$$

So

$$(e^{\cos x^2})' = y' = e^u \cdot (-\sin x^2)(2x) = (e^{\cos x^2}) \cdot (-\sin x^2)(2x)$$

Therefore

$$(\star) = f'(x) = (e^{\cos x^2}) \cdot (-\sin x^2)(2x) \sin x + e^{\cos x^2} (\cos x)$$

Question 2.(b).iii:

$$y = (\cos x)^x$$

$$\Rightarrow \ln y = \ln(\cos x)^x \Rightarrow \ln y = x \ln \cos x$$

$$\Rightarrow (\ln y)' = (x \ln \cos x)' \Rightarrow \frac{y'}{y} = 1 \cdot \ln \cos x + x \cdot \frac{(\cos x)'}{\cos x}$$

$$\Rightarrow \frac{y'}{y} = \ln \cos x - \frac{x \sin x}{\cos x}$$

$$\Rightarrow y' = y \left[ \ln \cos x - \frac{x \sin x}{\cos x} \right] = (\cos x)^x \left[ \ln \cos x - \frac{x \sin x}{\cos x} \right]$$