MA140-Engineering Calculus

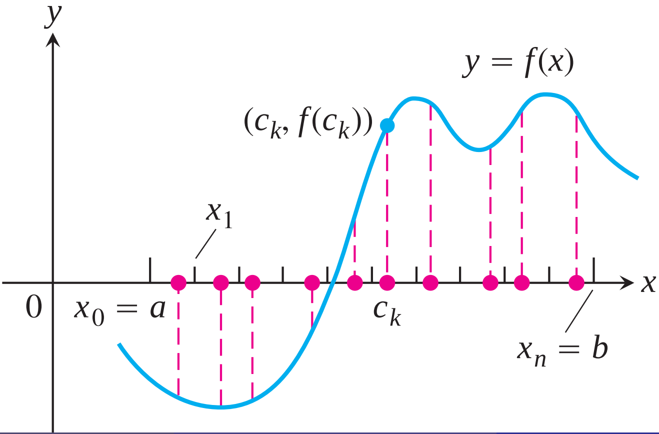
Lecture 33

November 16, 2017

Average value of a function over a range:

The average of n numbers is:

$$\frac{1}{n}\sum_{i=1}^{n} x_i$$



We divide $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. The average of the n sampled values is:

$$\frac{f(c_1) + f(c_2) + \cdots + f(c_n)}{n} = \frac{1}{n} \sum_{k=1}^{n} f(c_k)$$

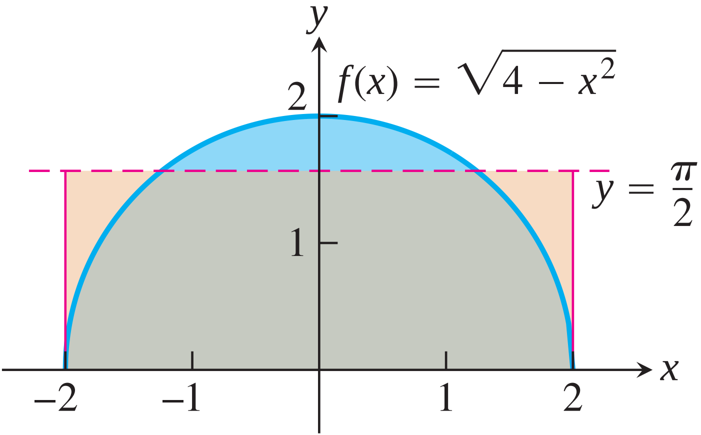
$$= \frac{\Delta x}{b - a} \sum_{k=1}^{n} f(c_k) = \frac{1}{b - a} \sum_{k=1}^{n} f(c_k) \Delta x$$

The average is obtained by dividing a Riemann sum for f on $[a, b]$ by $(b - a)$, so $n \to \infty$ or $\Delta x \to 0$, we get:

$$av(f) = \frac{1}{b - a} \int_{a}^{b} f(x) dx$$

Example 1.1

Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$



The average of the function is:

$$av(f) = \frac{1}{b-a}\int_a^b f(x)dx = \frac{Area}{b-a}$$

The area between the semicircle and the $x-$axis from -2 to 2 can be computed using the geometry formula:

$$Area = \frac{1}{2} \cdot \pi r^2 = \frac{1}{2} \cdot \pi (2)^2 = 2\pi$$

So the average value of f is

$$av(f) = \frac{1}{4}(2\pi) = \frac{\pi}{2}$$

It is sometimes convenient to use an alternative kind of average for the values of a function, $f(x)$, between $x = a$ and $x = b$.
The Root Mean Square Value provides a measure of central tendency for the numerical values of $f(x)$ and is defined to be the square root of the Mean Value of $f^2(x)$ from $x = a$ to $x = b$. Hence:

$$R.M.S = \sqrt{\frac{1}{b-a}\int_a^b [f(x)]^2 dx}$$

Example 1.2

An electric current $i(\theta)$ is given by $i(\theta) = I\sin(\theta)$ where I is a constant. Find R.M.S of $i(\theta)$ over $0 \leq \theta \leq 2\pi$

$$R.M.S = \sqrt{\frac{1}{b-a}\int_a^b [f(x)]^2 dx}$$

So

$$R.M.S = \sqrt{\frac{1}{2\pi - 0}\int_0^{2\pi} [I\sin\theta]^2 d\theta}$$

Then:

$$[R.M.S]^2 = \frac{1}{2\pi - 0}\int_0^{2\pi} [I\sin\theta]^2 d\theta = \frac{I^2}{2\pi}\int_0^{2\pi} \sin^2\theta d\theta$$

Recall $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$, so

$$[R.M.S]^2 = \frac{I^2}{2\pi}\int_0^{2\pi} \frac{1}{2}(1-\cos 2\theta)d\theta$$

So:

$$[R.M.S]^2 = \frac{I^2}{2\pi}\frac{1}{2}(\theta - \frac{1}{2}\sin 2\theta)\Big]_0^{2\pi}$$

Therefore

$$[R.M.S]^2 = \frac{I^2}{4\pi}\Big[(2\pi - \frac{1}{2}\sin(4\pi)) - (0 - \frac{1}{2}\sin(0))\Big]$$

Then:

$$[R.M.S]^2 = \frac{I^2}{4\pi}(2\pi) = \frac{I^2}{2} \Rightarrow R.M.S = \frac{I}{\sqrt{2}}$$

Example 1.3

Find the centroid of the area under the curve $y = 2x$ between the lines $x = 0$ and $x = 1$