Example 1.1

A robot "jumps" (vertically) on the moon, with initial velocity $3ms^{-1}$. Assuming zero (air) resistance and that gravity is one sixth of gravity on the earth, solve the differential equation $d^2x/dt^2 = -g/6$ to calculate

1. the height of the jump
2. how long the jump lasts for (the time difference between jumping and landing back in the same position)

($x(t)$ is the vertical height as a function of time.)

This is a second order linear differential equation of the form $\ddot{x} = -g/6$. We can integrate directly:

$$\int \ddot{x} dt = \int \left(\frac{-g}{6} \right) dt \implies \dot{x} = \frac{-gt}{6} + C_1$$

We integrate a second time:

$$\int \dot{x} dt = \int \left(\frac{-gt}{6} + C_1 \right) dt \implies x = \frac{-gt^2}{12} + C_1 t + C_2$$

The information given in the question can be used to fix C_1 and C_2: At time $t = 0$, the initial position is $x(0) = 0$ and the initial velocity is $\dot{x}(0) = 3$:

- $x(0) = 0 \implies C_2 = 0$
- $\dot{x}(0) = 3 \implies C_1 = 3$

so $x(t) = -gt^2/12 + 3t$. When $x(t) = 0$, the robot is on the ground, so

$$-gt^2/12 + 3t = 0 = t(-gt/12 + 3) \implies t = 0 \text{ or } 3 - gt/12 = 0$$

which gives $t = 36/g$, or just under 4 seconds.

At maximum height, the velocity is zero:

$$\frac{-gt}{6} + 3 = 0 \implies t = \frac{18}{g}$$

and at this time the position is

$$x(t) = x(18/g) = \left(\frac{18}{g}\right)\left(3 - \frac{18g}{12g}\right) = \left(\frac{18}{g}\right)(3 - \frac{3}{2}) = 27/g$$

(just a little under 3 metres).

Second Order Homogeneous Linear Differential Equations With Constant Coefficients

General linear differential equation (of order n):

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + a_{n-2}(x)y^{(n-2)} + \dots$$
$$\dots + a_2(x)y^{(2)} + a_1(x)y^{(1)} + a_0(x)y = b(x)$$

where

$$y^{(n)} = \frac{d^n y}{dx^n}$$

(differentiate n times).

1. **Second Order** means $n = 2$.
2. **Homogeneous** means $b(x) = 0$.
3. **Constant Coefficient** means all the "functions" $a_n(x), a_{n-1}(x), \dots, a_2(x), a_1(x), a_0(x)$ are just numbers (constants).

$$ay'' + by' + c = 0$$

The *auxilliary equation* (in variable m) is $am^2 + bm + c = 0$ which has solutions

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Depending on the solutions, we have three cases:

1. **Two identical roots** $b^2 - 4ac = 0$.
   Solution $y = (A + Bx)e^{mx}$
2. **Two different real roots** $b^2 - 4ac > 0$.
   Solution $y = Ae^{m_1 x} + Be^{m_2 x}$
3. **Two different complex roots** $b^2 - 4ac < 0$.
   Solution $y = e^{px}(A\cos(qx) + B\sin(qx))$ (where the roots are $p \pm iq$)

Newtons Laws (as differential equations)

1. **First Law** Every body continues in a state of rest or of uniform motion in a straight line unless acted on by an external force.
2. **Second Law** Force equals mass times acceleration.

In fact:

1. The Second Law tells us which differential equation to solve.
2. The First Law tells us a solution.

- $\ddot{x} = 0$ Uniform motion in a straight line
- $\ddot{x} = g$ Acceleration due to (constant) gravity
- $\ddot{x} = -kx$ Motion of a Spring (Force varies (linearly) with position)