Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and $g'(a) \neq 0$. Then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Proof: Working backward from $f'(a)$ and $g'(a)$, which are themselves limits, we have

$$\frac{f'(a)}{g'(a)} = \frac{\lim_{x \to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x \to a} \frac{g(x)-g(a)}{x-a}} = \lim_{x \to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{f(x) - 0}{g(x) - 0} = \lim_{x \to a} \frac{f(x)}{g(x)}$$

Theorem 1.3

L'Hôpital's Rule:

If f *and* g *are differentiable and* $g'(x) \neq 0$ *near* a *(except possibly at* a*),*

- *if* $\lim_{x\to a} f(x) = 0$ *and* $\lim_{x\to a} g(x) = 0$
  or
- $\lim_{x\to a} f(x) = \pm\infty$ *and* $\lim_{x\to a} g(x) = \pm\infty$

Then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Example 1.4

Find the following limit:

$$\lim_{x \to 1} \frac{\ln x}{x - 1}$$

As $\lim_{x\to 1}(\ln x) = \ln(1) = 0$ and $\lim_{x\to 1}(x-1) = 0$, we can apply L'Hôpital's Rule:

$$\lim_{x \to 1} \frac{\ln x}{x - 1} \overset{\text{H}}{=} \lim_{x \to 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x \to 1} \frac{\frac{1}{x}}{1} = \lim_{x \to 1} \frac{1}{x} = \frac{1}{1} = 1$$

Example 1.5

Find
$$\lim_{x\to\infty} \frac{e^x}{x^2}$$

$\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty} x^2 = \infty$, so we can apply the L'Hôpital's Rule:
$$\lim_{x\to\infty} \frac{e^x}{x^2} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{2x}$$

This result is also of the form $\frac{\infty}{\infty}$, so we can apply the L'Hôpital's Rule again:
$$\lim_{x\to\infty} \frac{e^x}{2x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

Example 1.6

Find

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3}$$

$\lim_{x\to 0}(\sin(x) - x) = 0$ and $\lim_{x\to 0} x^3 = 0$, as we see the limit is of the form $\frac{0}{0}$. so

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3} \overset{\text{H}}{=} \lim_{x\to 0} \frac{\cos(x) - 1}{3x^2}$$

Note that $\lim_{x\to 0}(\cos x - 1) = 0$ and $\lim_{x\to 0}(3x^2) = 0$, so again we get $\frac{0}{0}$, so we can to apply L'Hôpital's Rule again,

$$\lim_{x\to 0} \frac{\cos(x) - 1}{3x^2} \overset{\text{H}}{=} \lim_{x\to 0} \frac{-\sin x}{6x} = \frac{0}{0}$$

We can use L'Hôpital's Rule again:

$$\lim_{x\to 0} \frac{-\sin x}{6x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{-\cos x}{6} = \frac{-1}{6}$$