MA140-Engineering Calculus

Lecture 20

October 29, 2019

Example 1.1

Evaluate:

$$\int \frac{dx}{x(x^2+1)^2}$$

The form of the partial fraction decomposition is

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Add up the fractions:

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$= \frac{A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x}{x(x^2+1)^2}$$

So:

$$1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$$
$$\Rightarrow 1 = A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex$$
$$\Rightarrow 1 = (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

If we equate coefficients, we get the system

$$\begin{cases} A+B &= 0 \\ C &= 0 \\ 2A+B+D &= 0 \\ C+E &= 0 \\ A &= 1 \end{cases}$$

Solving this system gives: $A = 1, B = -1, C = 0, D = -1, E = 0$
Thus

$$\int \frac{dx}{x(x^2+1)^2} = \int \Big(\frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}\Big)dx$$

$$= \int \Big(\frac{1}{x} + \frac{-x+0}{x^2+1} + \frac{-x+0}{(x^2+1)^2}\Big)dx$$

$$= \int \Big(\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}\Big)dx$$

$$= \underbrace{\int \frac{1}{x}dx}_{I_1} + \underbrace{\int \frac{-x}{x^2+1}dx}_{I_2} + \underbrace{\int \frac{-x}{(x^2+1)^2}dx}_{I_3}$$

$$I_1 = \int \frac{1}{x} dx$$

Therefore

$$I_1 = \ln(x) + c_1$$

Also

$$I_2 = \int \frac{-x}{x^2+1} dx$$

Let $u = x^2 + 1$ then $du = 2xdx$, so $-\frac{du}{2} = -xdx$

$$\int \frac{-x}{x^2+1} dx = \int \frac{-du}{2u} = \frac{-1}{2} \int \frac{du}{u} = \frac{-1}{2} \ln u + c_2 = \frac{-1}{2} \ln(x^2+1) + c_2$$

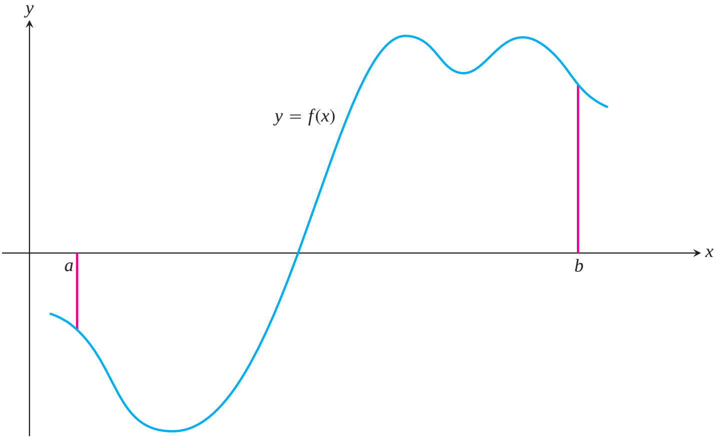
$$I_3 = \int \frac{-x}{(x^2+1)^2} dx$$

Let $u = x^2 + 1$, then $du = 2xdx$, so $\frac{-du}{2} = -xdx$, Therefore:

$$I_3 = \int \frac{-x}{(x^2+1)^2} dx = \int \frac{-du}{2(u)^2} = \frac{-1}{2} \int \frac{du}{u^2} = \frac{-1}{2} \int u^{-2} du$$

$$= \frac{-1}{2}\left(\frac{u^{-2+1}}{-2+1}\right) + c_3 = \frac{-1}{2}\left(\frac{(x^2+1)^{-2+1}}{-2+1}\right) + c_3 = \frac{-1}{2}\left(\frac{(x^2+1)^{-1}}{-1}\right) + c_3$$

Definite Integral as area under the graph



We choose $n-1$ points $\{x_1, x_2, \cdots, x_{n-1}\}$ between a and b and satisfying

$$a < x_1 < x_2 < \cdots < x_{n-1} < b$$

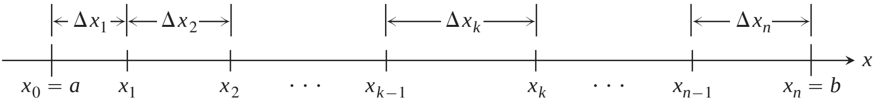
To make the notation consistent, we denote a by x_0 and b by x_n, so that :

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

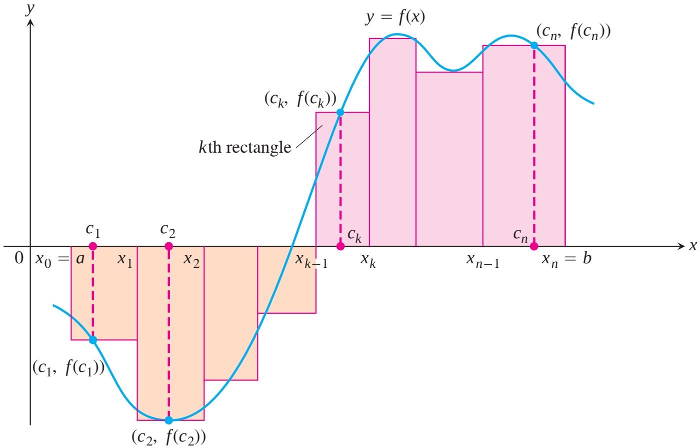
The set

$$P = \{x_0, x_1, x_2, \cdots, x_{n-1}, x_n\}$$

is called a partition of $[a, b]$

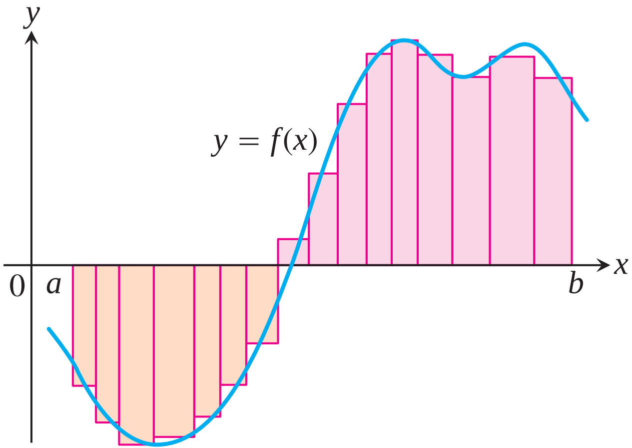


In each subinterval we select some point c_k. Then on each subinterval we stand a vertical rectangle that stretches from the x-axis to touch the curve at $(c_k, f(c_k))$:



We can also let all the the subintervals to have equal widths, $\Delta x_k = \frac{b-a}{n}$
Then the area of each rectangle is

$$\Delta x_k \cdot f(c_k) = \frac{b-a}{n} \cdot f(c_k)$$

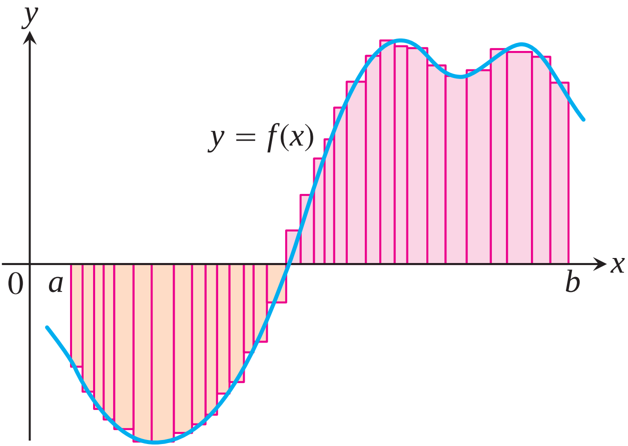


Finally we sum all these products to get

$$S_P = \sum_{k=1}^{n} \frac{b-a}{n} \cdot f(c_k) = \frac{b-a}{n} \sum_{k=1}^{n} f(c_k)$$

S_P is called a Riemann sum for f on the interval $[a, b]$.

We can also make the width of the subintervals smaller.

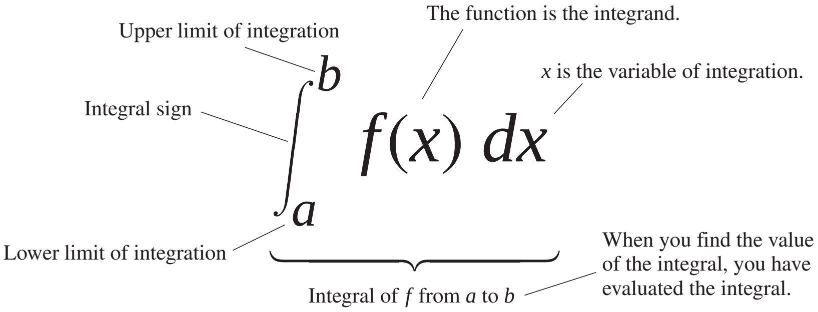


If we make n larger (or make $\frac{1}{n}$ smaller), then

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{b-a}{n}\cdot f(c_k)$$

will give us an accurate answer.
We say that the definite integral of f from a to b is:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{b-a}{n}\right)\cdot f(c_k)$$


The function is the integrand.
Upper limit of integration
x is the variable of integration.
Integral sign
$\int_a^b f(x)\,dx$
Lower limit of integration
When you find the value of the integral, you have evaluated the integral.
Integral of f from a to b

Theorem 1.2

The Fundamental Theorem of Calculus (1):

Suppose that f is a continuous function on $[a, b]$, and F is an anti-derivative of f. Then:

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Example 1.3

Evaluate

$$\int_0^1 x^3 dx$$

An anti-derivative can be obtained by evaluating the following integral:

$$\int x^3 dx = x^4/4$$

So $F(x) = x^4/4$, therefore:

$$\int_0^1 x^3 dx = \left[x^4/4\right]_0^1 = (1^4/4) - (0^4/4) = 1/4$$

Theorem 1.4

- *Order of integration:*

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

- *Zero width interval*

$$\int_a^a f(x)dx = 0$$

- *if* $a < c < b$*, then*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Example 1.5

Suppose that $\int_{-1}^{1} f(x)dx = 5$, $\int_{1}^{4} f(x)dx = -2$ and $\int_{-1}^{1} h(x)dx = 7$. Then find

- $$\int_{4}^{1} f(x)dx$$
- $$\int_{-1}^{1} [2f(x) + 3h(x)]dx$$
- $$\int_{-1}^{4} f(x)dx$$

Theorem 1.6

The Fundamental Theorem of Calculus (2):

Suppose that f is a continuous function on $[a, b]$,

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Example 1.7

Find

$$\frac{d}{dx}\int_0^x \sqrt{1+2t}dt$$

Recall that

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

so:

$$\frac{d}{dx}\int_0^x \sqrt{1+2t}dt = \sqrt{1+2x}$$

Example 1.8

Find $\frac{dy}{dx}$, if

$$y = \int_a^x \cos(t)dt$$

By The fundamental theorem of calculus, we know that:

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

So:

$$\frac{d}{dx}\int_a^x \cos(t)dt = \cos(x)$$