MA140-Engineering Calculus

Lecture 14

October 14, 2019

Example 1.1

sketch a graph of the function $f(x) = x^4 - 4x^3 + 10$

We use the following steps:

(1) Find the critical (stationary) points

(2) Find the points of inflection

(3) Use the second derivative test

(4) Find the y-value of these points to pair them

(5) Draw the table to find the intervals on which f is increasing and the intervals on which f is decreasing

(6) Add some extra rows to your table to see where the graph of f is concave up and where it is concave down

(7) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts.

(8) Sketch the general shape of the graph for f

Step (1):

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) = 0$$

So the stationary points are $x = 0$ and $x = 3$

Step (2):

$$f''(x) = 12x^2 - 24x = 12x(x-2) = 0$$

So the points of inflection are $x = 0$ and $x = 2$

Step (3):

At $x = 0$, $f''(0) = 0$ so the test fails in this point. But at $x = 3$, $f''(3) = 36 > 0$ so based on the test, $x = 3$ is a local minimum

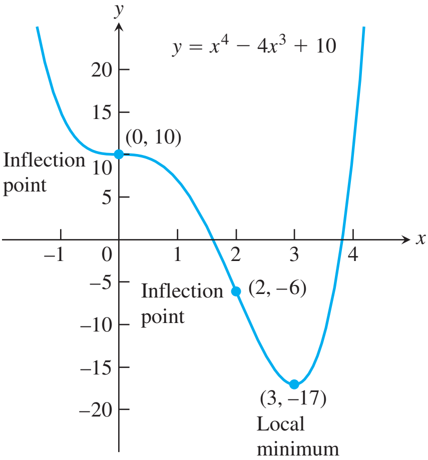
Step 4:

$f(0) = 10$, $f(2) = -6$ and $f(3) = -17$

Step 5 and 6:

		0		2		3	
$4x^2$	$+$	•	$+$		$+$		$+$
$x-3$	$-$		$-$		$-$	•	$+$
$f'(x)$	$-$	•	$-$		$-$	•	$+$
$12x$	$-$	•	$+$		$+$		$+$
$x-2$	$-$		$-$	•	$+$		$+$
$f''(x)$	$+$	•	$-$	•	$+$		$+$

Step 7 and 8:


y
y = x^4 − 4x^3 + 10
20
15
(0, 10)
Inflection
point
10
5
x
–1
0
1
2
3
4
–5
Inflection
point
(2, –6)
–10
–15
–20
(3, –17)
Local
minimum

Example 1.2

Sketch the graph of

$$f(x) = \frac{(x+1)^2}{1+x^2}$$

Step (1):

$f(x) = \dfrac{(x+1)^2}{1+x^2}$, so

$$f'(x) = \frac{(1+x^2)\cdot 2(x+1) - (x+1)^2 \cdot 2x}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2}$$

So the critical points are $x = -1$ and $x = 1$.

Step (2):

$$f''(x) = \frac{(1+x^2)^2 \cdot 2(-2x) - 2(1-x^2)[2(1+x^2)\cdot 2x]}{(1+x^2)^4} = \frac{4x(x^2-3)}{(1+x^2)^3}$$

So inflection points are $x = \sqrt{3}$, $x = 0$ and $x = -\sqrt{3}$

Step (3):

since $f''(1) = -1 < 0$ so $x = 1$ is a local max and since $f''(-1) = 1 > 0$ so $x = -1$ is a local min

Step (4):

$f(1) = 2,\ f(-1) = 0,\ f(\sqrt{3}) = \frac{(\sqrt{3}+1)^2}{4},\ f(-\sqrt{3}) = \frac{(-\sqrt{3}+1)^2}{4}, f(0) = 1$

We can also find the asymptotes of the function:

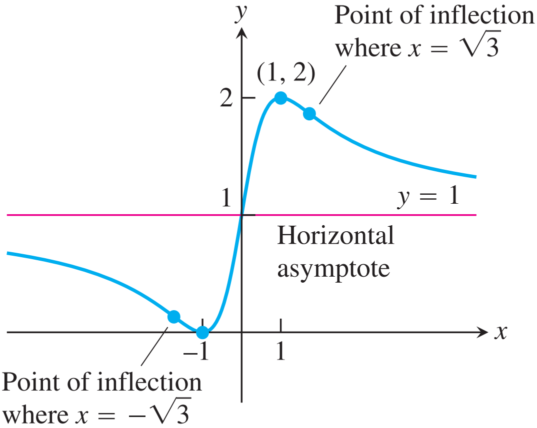
$$\lim_{x\to\infty} f(x) = 1$$

So $y = 1$ is the horizontal asymptote.

Step 5 and 6:

		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
$2(1-x^2)$	$-$		$-$	•	$+$		$+$	•	$-$		$-$
$(1+x^2)^2$	$+$		$+$		$+$		$+$		$+$		$+$
$f'(x)$	$-$		$-$	•	$+$		$+$	•	$-$		$-$
$4x$	$-$		$-$		$-$	•	$+$		$+$		$+$
(x^2-3)	$+$	•	$-$		$-$		$-$		$-$	•	$+$
$(1+x^2)^3$	$+$		$+$		$+$		$+$		$+$		$+$
$f''(x)$	$-$	•	$+$		$+$	•	$-$		$-$	•	$+$

Step 7 and 8:


y
Point of inflection
where $x = \sqrt{3}$
(1, 2)
2
1
$y = 1$
Horizontal
asymptote
x
−1
1
Point of inflection
where $x = -\sqrt{3}$

Exercise

Use the steps of the graphing procedure to graph the following equations:

- $$y = x^2 - 4x + 3$$
- $$y = x^3 - 3x + 3$$
- $$y = \frac{x^2 - 3}{x - 2}$$